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Numerical approximation to a solution of the modified regularized long wave equation using quintic B-splines

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Abstract

In this work, a numerical solution of the modified regularized long wave (MRLW) equation is obtained by the method based on collocation of quintic B-splines over the finite elements. A linear stability analysis shows that the numerical scheme based on Von Neumann approximation theory is unconditionally stable. Test problems including the solitary wave motion, the interaction of two and three solitary waves and the Maxwellian initial condition are solved to validate the proposed method by calculating error norms L_2 and L_∞ that are found to be marginally accurate and efficient. The three invariants of the motion have been calculated to determine the conservation properties of the scheme. The obtained results are compared with other earlier results.

MSC: 97N40; 65N30; 65D07; 76B25; 74S05

Keywords: MRLW equation; collocation; finite element method; B-spline; solitary waves

1 Introduction

The modified regularized long wave (MRLW) equation, based upon the regularized long wave (RLW) equation,

$$U_t + U_x + \delta U U_x - \mu U_{xxt} = 0, \quad (1)$$

which was proposed at first by Peregrine [1] to describe the development of an undular bore, has the form

$$U_t + U_x + 6U^2 U_x - \mu U_{xxt} = 0, \quad (2)$$

where δ and μ are positive parameters and the subscripts x and t denote the differentiation. The RLW equation is one of the best known partial differential equations because it describes a large number of important physical phenomena with weak nonlinearity and dispersion waves such as magneto hydrodynamic and ion-acoustic waves in plasma, phonon packets in non-linear crystals, the transverse waves in shallow water, rotating flow down a tube and pressure waves in liquid-gas bubble mixtures. Bona and Pryant [2] have studied the existence and uniqueness of the equation. Benjamin *et al.* [3] have proposed the RLW equation as a numerically superior modification of the Korteweg de-Vries (KdV) equation.

This superiority arises because, unlike the KdV equation, the dispersion relation associated with the linearized RLW equation yields the frequency that is bounded for large wave numbers [4]. But they have found an analytical solution of the RLW equation under the restricted initial and boundary conditions. So, various numerical techniques have been introduced to solve the equation. These include the finite difference [5–7], finite element [8–22], Fourier pseudo-spectral [23] methods and the meshfree method [24]. One of the special properties of the equation is that the solutions may exhibit solitons whose magnitudes, shapes and velocities are not changed after the collision. The RLW equation is a special case of the generalized long wave (GRLW) equation having the form

$$U_t + U_x + \delta U^p U_x - \mu U_{xxt} = 0, \quad (3)$$

where p is a positive integer. Zhang [25] has used the finite difference method to solve the GRLW equation for a Cauchy problem. The quasilinearization method based on finite differences was used by Ramos [26] for solving the GRLW equation. Kaya *et al.* [27] have also studied the GRLW equation with the Adomian decomposition method. Roshan [28] has solved the GRLW equation numerically by the Petrov-Galerkin method using a linear hat function as the trial function and a quintic B-spline function as the test function. Gardner *et al.* [29] have developed a collocation solution to the MRLW equation using quintic B-splines finite elements. Khalifa *et al.* [30, 31] have obtained the numerical solutions of the MRLW equation using the finite difference method and the cubic B-spline collocation finite element method. Solutions based on the collocation method with quadratic B-spline finite elements and the central finite difference method for time have been investigated by Raslan [32]. Raslan and Hassan [33] have solved the MRLW equation by the collocation finite element method using quadratic, cubic, quartic and quintic B-splines to obtain the numerical solutions of a single solitary wave. Fazal-i-Haq *et al.* [34] have designed a numerical scheme based on the quartic B-spline collocation method for the numerical solution of the MRLW equation. Ali [35] has formulated a classical radial basis functions (RBFs) collocation method for solving the MRLW equation. In this paper, we have obtained a type of the quintic B-spline collocation procedure in which a nonlinear term in the equation is linearized by using the form introduced by Rubin and Graves [36] to solve the MRLW equation. The proposed method is shown to represent accurately the migration of a single solitary wave. Then the interaction of two and three solitary waves and the Maxwellian initial condition are studied. The linear stability analysis based on the Von Neumann method is also investigated.

2 Quintic B-spline finite element solution

Let us consider MRLW equation (2) with the following initial,

$$U(x, 0) = f(x), \quad a \leq x \leq b, \quad (4)$$

and boundary conditions:

$$\begin{aligned} U(a, t) &= 0, & U(b, t) &= 0, \\ U_x(a, t) &= 0, & U_x(b, t) &= 0, \\ U_{xx}(a, t) &= 0, & U_{xx}(b, t) &= 0, \quad t > 0. \end{aligned} \quad (5)$$

For the numerical calculation, the solution domain of the problem is restricted over an interval $a \leq x \leq b$. The interval is partitioned into uniformly-sized finite elements of length h by the knots x_m such that $a = x_0 < x_1 < \dots < x_N = b$. The set of quintic B-spline functions $\{\phi_{-2}(x), \phi_{-1}(x), \dots, \phi_{N+1}(x), \phi_{N+2}(x)\}$ forms a basis over the problem domain $[a, b]$. We seek the numerical solution $U_N(x, t)$ to the exact solution $U(x, t)$ in the form of

$$U_N(x, t) = \sum_{j=-2}^{N+2} \phi_j(x) \delta_j(t), \tag{6}$$

where $\delta_j(t)$ are time dependent parameters to be determined from the boundary and collocation conditions.

Quintic B-splines $\phi_m(x)$ ($m = -2(1)N + 2$), at the knots x_m are defined over the interval $[a, b]$ by [37].

$$\phi_m(x) = \frac{1}{h^5} \begin{cases} (x - x_{m-3})^5, & [x_{m-3}, x_{m-2}], \\ (x - x_{m-3})^5 - 6(x - x_{m-2})^5, & [x_{m-2}, x_{m-1}], \\ (x - x_{m-3})^5 - 6(x - x_{m-2})^5 + 15(x - x_{m-1})^5, & [x_{m-1}, x_m], \\ (x - x_{m-3})^5 - 6(x - x_{m-2})^5 + 15(x - x_{m-1})^5 - 20(x - x_m)^5, & [x_m, x_{m+1}], \\ (x - x_{m-3})^5 - 6(x - x_{m-2})^5 + 15(x - x_{m-1})^5 - 20(x - x_m)^5 + 15(x - x_{m+1})^5, & [x_{m+1}, x_{m+2}], \\ (x - x_{m-3})^5 - 6(x - x_{m-2})^5 + 15(x - x_{m-1})^5 - 20(x - x_m)^5 + 15(x - x_{m+1})^5 - 6(x - x_{m+2})^5, & [x_{m+2}, x_{m+3}], \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

Each quintic B-spline covers six elements so that each element $[x_m, x_{m+1}]$ is covered by six B-splines. Substituting trial function (7) into Eq. (6), the nodal values of U, U', U'' at the knots x_m are obtained in terms of the element parameters δ_m by

$$\begin{aligned} U_N(x_m, t) &= U_m = \delta_{m-2} + 26\delta_{m-1} + 66\delta_m + 26\delta_{m+1} + \delta_{m+2}, \\ U'_m &= \frac{5}{h}(-\delta_{m-2} - 10\delta_{m-1} + 10\delta_{m+1} + \delta_{m+2}), \\ U''_m &= \frac{20}{h^2}(\delta_{m-2} + 2\delta_{m-1} - 6\delta_m + 2\delta_{m+1} + \delta_{m+2}), \end{aligned} \tag{8}$$

where the symbols ' and '' represent first and second differentiation with respect to x , respectively. The splines $\phi_m(x)$ and their four principle derivatives vanish outside the interval $[x_{m-3}, x_{m+3}]$.

Using a first-order forward difference formula for the time derivative of the U and Crank-Nicolson approximation for the space derivatives U_x and U_{xx} in Eq. (2) leads to

$$\frac{U^{n+1} - U^n}{\Delta t} + \frac{U_x^{n+1} + U_x^n}{2} + 6 \frac{(U^2 U_x)^{n+1} + (U^2 U_x)^n}{2} - \mu \frac{U_{xx}^{n+1} - U_{xx}^n}{\Delta t} = 0. \tag{9}$$

Now, if we apply a linearization technique similar to the one first introduced by Rubin and Graves [36] to Eq. (9),

$$(U^2 U_x)^{n+1} = U^{n+1} U^n U_x^n + U^n U^{n+1} U_x^n + U^n U^n U_x^{n+1} - 2U^n U^n U_x^n,$$

we obtain

$$\begin{aligned}
 & U^{n+1} + \frac{\Delta t}{2} U_x^{n+1} + 3\Delta t(U^{n+1}U^n U_x^n + U^n U^{n+1}U_x^n + U^n U^n U_x^{n+1}) - \mu U_{xx}^{n+1} \\
 & = U^n - \frac{\Delta t}{2} U_x^n - 3\Delta t(U^2 U_x)^n - \mu U_{xx}^n + 6\Delta t(U^n U^n U_x^n).
 \end{aligned}$$

If we substitute the nodal values of U , U_x and U_{xx} given by (8) into (10), we obtain the following iterative system:

$$\begin{aligned}
 & \delta_{m-2}^{n+1}(1 - \alpha_1 + 2\alpha_2 - \alpha_3 - \alpha_4) + \delta_{m-1}^{n+1}(26 - 10\alpha_1 + 52\alpha_2 - 10\alpha_3 - 2\alpha_4) \\
 & + \delta_m^{n+1}(66 + 132\alpha_2 + 6\alpha_4) + \delta_{m+1}^{n+1}(26 + 10\alpha_1 + 52\alpha_2 + 10\alpha_3 - 2\alpha_4) \\
 & + \delta_{m+2}^{n+1}(1 + \alpha_1 + 2\alpha_2 + \alpha_3 - \alpha_4) \\
 & = \delta_{m-2}^n(1 + \alpha_1 + \alpha_2 - \alpha_4) + \delta_{m-1}^n(26 + 10\alpha_1 + 26\alpha_2 - 2\alpha_4) + \delta_m^n(66 + 66\alpha_2 + 6\alpha_4) \\
 & + \delta_{m+1}^n(26 - 10\alpha_1 + 26\alpha_2 - 2\alpha_4) + \delta_{m+2}^n(1 - \alpha_1 + \alpha_2 - \alpha_4), \quad m = 0(1)N, \quad (10)
 \end{aligned}$$

where

$$\begin{aligned}
 \alpha_1 & = \frac{5\Delta t}{2h}, \\
 \alpha_2 & = \frac{15\Delta t}{h}(\delta_{m-2}^n + 26\delta_{m-1}^n + 66\delta_m^n + 26\delta_{m+1}^n + \delta_{m+2}^n)(-\delta_{m-2}^n - 10\delta_{m-1}^n + 10\delta_{m+1}^n + \delta_{m+2}^n), \\
 \alpha_3 & = \frac{15\Delta t}{h}(\delta_{m-2}^n + 26\delta_{m-1}^n + 66\delta_m^n + 26\delta_{m+1}^n + \delta_{m+2}^n)^2, \\
 \alpha_4 & = \frac{20\mu}{h^2}.
 \end{aligned}$$

This newly obtained iterative system (10) consists of $N + 1$ linear equations including $N + 5$ unknown parameters $(\delta_{-2}, \delta_{-1}, \dots, \delta_{N+1}, \delta_{N+2})^T$. To obtain a unique solution to this system, we need four additional constraints. These are obtained from the boundary conditions $U(a, t) = U(b, t) = 0$ and $U_x(a, t) = U_x(b, t) = 0$ and can be used to eliminate δ_{-2} , δ_{-1} and δ_{N+1} , δ_{N+2} from system (10), which then becomes a matrix equation for the $N + 1$ unknowns $\mathbf{d} = (\delta_0, \delta_1, \dots, \delta_N)^T$ of the form

$$\mathbf{A}\mathbf{d}^{n+1} = \mathbf{B}\mathbf{d}^n. \quad (11)$$

The matrices \mathbf{A} and \mathbf{B} are pentagonal $(N + 1) \times (N + 1)$ matrices given as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & & & & & & & \\ a_{21} & a_{22} & a_{23} & a_{24} & & & & & & \\ & & \ddots & & & & & & & \\ & a_{m,m-2} & a_{m,m-1} & a_{m,m} & a_{m,m+1} & a_{m,m+2} & & & & \\ & & & & \ddots & & & & & \\ & & & & a_{n,n-2} & a_{n,n-1} & a_{n,n} & a_{n,n+1} & & \\ & & & & & a_{n+1,n-1} & a_{n+1,n} & a_{n+1,n+1} & & \end{bmatrix}, \quad m = 3(1)n - 1,$$

and δ_{N+1} :

$$\begin{aligned}
 U_N(x, 0) &= U(x_m, 0), \quad m = 0, 1, \dots, N, \\
 (U_N)_x(a, 0) &= 0, \quad (U_N)_x(b, 0) = 0, \\
 (U_N)_{xx}(a, 0) &= 0, \quad (U_N)_{xx}(b, 0) = 0,
 \end{aligned}
 \tag{13}$$

we obtain the following matrix form for the initial vector \mathbf{d}^0 :

$$\mathbf{W} \mathbf{d}^0 = \mathbf{b}, \tag{14}$$

where

$$\mathbf{W} = \begin{bmatrix} 54 & 60 & 6 & & & & & & & \\ 25.25 & 67.50 & 26.25 & 1 & & & & & & \\ & 1 & 26 & 66 & 26 & 1 & & & & \\ & & 1 & 26 & 66 & 26 & 1 & & & \\ & & & & \ddots & & & & & \\ & & & & & 1 & 26 & 66 & 26 & 1 \\ & & & & & & 1 & 26.25 & 67.50 & 25.25 \\ & & & & & & & 6 & 60 & 54 \end{bmatrix},$$

$$\mathbf{d}^0 = (\delta_0, \delta_1, \delta_2, \dots, \delta_{N-2}, \delta_{N-1}, \delta_N)^T$$

and

$$\mathbf{b} = (U(x_0, 0), U(x_1, 0), U(x_2, 0), \dots, U(x_{N-2}, 0), U(x_{N-1}, 0), U(x_N, 0))^T.$$

2.1 A linear stability analysis

The stability analysis is based on the Von Neumann theory in which the growth factor of a typical Fourier mode is defined as

$$\delta_j^n = \widehat{\zeta}^n e^{ijkh}, \tag{15}$$

where k is a mode number and h is the element size. The non-linear term $U^2 U_x$ of the MRLW equation cannot be handled by the Fourier mode method. Thus, this term is linearized by making the quantity U^2 in the nonlinear term a local constant such as Z_m . Then substituting Eq. (15) into system (10) gives

$$\widehat{\zeta}^{n+1} = g \widehat{\zeta}^n, \tag{16}$$

where g is the growth factor.

Now, we identify the collocation points with the knots and use Eq. (8) to evaluate U_m and its necessary space derivatives and substitute into Eq. (2) to obtain the following equa-

tion:

$$\begin{aligned} & \dot{\delta}_{m-2} + 26\dot{\delta}_{m-1} + 66\dot{\delta}_m + 26\dot{\delta}_{m+1} + 26\dot{\delta}_{m+2} \\ & + \frac{5}{h}(1 + 6Z_m)(-\delta_{m-2} - 10\delta_{m-1} + 10\delta_{m+1} + \delta_{m+2}) \\ & - \frac{20\mu}{h^2}(\dot{\delta}_{m-2} + 2\dot{\delta}_{m-1} - 6\dot{\delta}_m + 2\dot{\delta}_{m+1} + 26\dot{\delta}_{m+2}) = 0. \end{aligned} \tag{17}$$

Here $\dot{\cdot}$ denotes derivative with respect to time. If time parameters δ_i 's and their time derivatives $\dot{\delta}_i$'s in Eq. (17) are discretized by the Crank-Nicolson formula and usual forward finite difference approximation, respectively:

$$\delta_i = \frac{\delta^n + \delta^{n+1}}{2}, \quad \dot{\delta}_i = \frac{\delta^{n+1} - \delta^n}{\Delta t}, \tag{18}$$

we obtain a recurrence relationship between two time levels n and $n + 1$ relating two unknown parameters $\delta_i^{n+1}, \delta_i^n$ for $i = m - 2, m - 1, \dots, m + 1, m + 2$

$$\begin{aligned} & \gamma_1\delta_{m-2}^{n+1} + \gamma_2\delta_{m-1}^{n+1} + \gamma_3\delta_m^{n+1} + \gamma_4\delta_{m+1}^{n+1} + \gamma_5\delta_{m+2}^{n+1} \\ & = \gamma_5\delta_{m-2}^n + \gamma_4\delta_{m-1}^n + \gamma_3\delta_m^n + \gamma_2\delta_{m+1}^n + \gamma_1\delta_{m+2}^n, \end{aligned} \tag{19}$$

where

$$\begin{aligned} \gamma_1 &= (1 - E - M), & \gamma_2 &= (26 - 10E - 2M), \\ \gamma_3 &= (66 + 6M), & \gamma_4 &= (26 + 10E - 2M), \\ \gamma_5 &= (1 + E - M), \\ m &= 0, 1, \dots, N, & E &= (1 + 6Z_m)\frac{5}{2h}\Delta t, & M &= \frac{20}{h^2}\mu. \end{aligned} \tag{20}$$

Substituting the Fourier mode (15) into (19) gives the growth factor g of the form

$$g = \frac{a - ib}{a + ib}, \tag{21}$$

where

$$\begin{aligned} a &= 33 + 3M + (26 - 2M)\cos[hk] + (1 - M)\cos[2hk], \\ b &= 10E\sin[hk] + E\sin[2hk]. \end{aligned} \tag{22}$$

The modulus of $|g|$ is 1, therefore the linearized scheme is unconditionally stable.

3 Results and discussion

In this section, we consider the following four test problems: the motion of a single solitary wave, the interaction of two and three solitary waves and the Maxwellian initial condition. Accuracy and efficiency of the method are measured by the error norms L_2

$$L_2 = \|U^{\text{exact}} - U_N\|_2 \simeq \sqrt{h \sum_{j=1}^N |U_j^{\text{exact}} - (U_N)_j|^2},$$

and L_∞

$$L_\infty = \|U^{\text{exact}} - U_N\|_\infty \simeq \max_j |U_j^{\text{exact}} - (U_N)_j|, \quad j = 1, 2, \dots, N - 1.$$

The MRLW equation satisfies only three conservation laws given by [38]

$$I_1 = \int_a^b U \, dx \simeq h \sum_{j=1}^N U_j^n,$$

$$I_2 = \int_a^b [U^2 + \mu(U_x)^2] \, dx \simeq h \sum_{j=1}^N [(U_j^n)^2 + \mu(U_x)_j^n],$$

$$I_3 = \int_a^b (U^4 - \mu U_x^2) \, dx \simeq h \sum_{j=1}^N [(U_j^n)^4 - \mu(U_x)_j^n],$$

which correspond to conservation of mass, momentum and energy, respectively. In the simulation of a solitary wave motion, the invariants I_1 , I_2 and I_3 are monitored to check the conservation of the numerical algorithm.

3.1 The motion of a single solitary wave

For this problem, MRLW Eq. (2) is considered with the boundary condition $U \rightarrow 0$ as $x \rightarrow \pm\infty$ and the initial condition

$$U(x, 0) = \sqrt{c} \operatorname{sech}(p(x - x_0)).$$

Note that the analytical solution of this problem can be written as

$$U(x, t) = \sqrt{c} \operatorname{sech}(p(x - (c + 1)t - x_0)),$$

where $p = \sqrt{\frac{c}{\mu(c+1)}}$, x_0 and c are arbitrary constants. The constants of motion, for a solitary wave of amplitude \sqrt{c} and width depending on p may be evaluated analytically as [29]

$$I_1 = \int_{-\infty}^{\infty} U(x, 0) \, dx = \frac{\pi \sqrt{c}}{p},$$

$$I_2 = \int_{-\infty}^{\infty} (U^2(x, 0) + \mu U_x^2(x, 0)) \, dx = \frac{2c}{p} + \frac{2\mu pc}{3}, \tag{23}$$

$$I_3 = \int_{-\infty}^{\infty} (U^4(x, 0) - \mu U_x^2(x, 0)) \, dx = \frac{4c^2}{3p} - \frac{2\mu pc}{3}.$$

For our computational work, we have chosen two sets of parameters. Firstly, we have used the parameters $c = 1$, $\mu = 1$, $h = 0.2$, $x_0 = 40$, $k = 0.025$ over the interval $[0, 100]$ to coincide with those of earlier papers [28–30, 35]. So, the solitary wave has amplitude 1.0 and the computations are done up to time $t = 10$ to obtain the invariants and error norms I_2 and L_∞ at various times. Error norms I_2 , L_∞ and three invariants of the MRLW equation are listed in Table 1. It is seen that the error norms are found to be small enough

**Table 1 Invariants and error norms for a single solitary wave with $c = 1, h = 0.2, k = 0.025,$
 $0 \leq x \leq 100$**

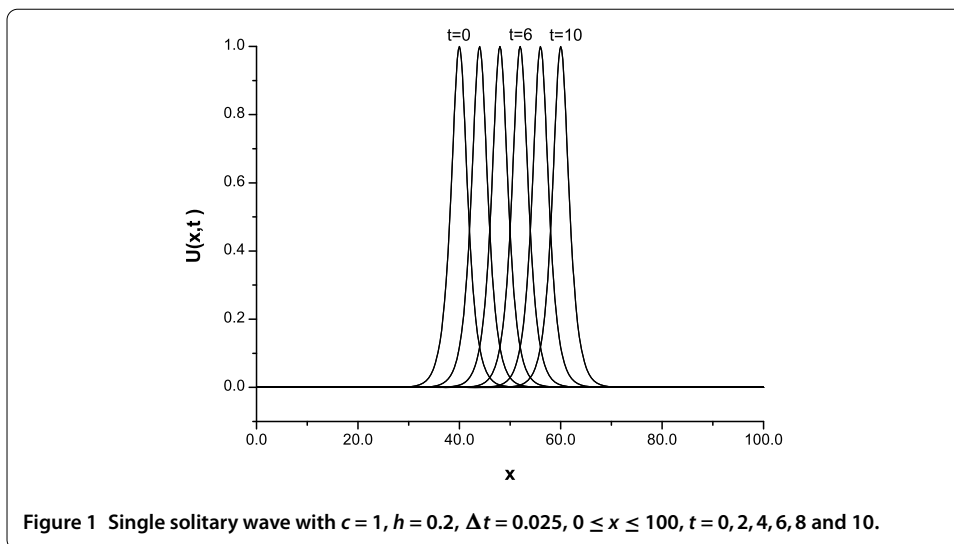
t	I_1	I_2	I_3	$L_2 \times 10^3$	$L_\infty \times 10^3$
0	4.4428660	3.2998226	1.4142046	0.00000000	0.00000000
1	4.4428660	3.2998068	1.4142204	0.28867055	0.17189210
2	4.4428660	3.2997776	1.4142496	0.56818930	0.32423631
3	4.4428660	3.2997536	1.4142736	0.83577886	0.46169463
4	4.4428660	3.2997375	1.4142897	1.09458149	0.59234819
5	4.4428660	3.2997272	1.4143000	1.34807894	0.72040575
6	4.4428660	3.2997206	1.4143066	1.59852566	0.84732049
7	4.4428660	3.2997164	1.4143108	1.84722430	0.97368288
8	4.4428660	3.2997137	1.4143135	2.09491698	1.09976163
9	4.4428661	3.2997119	1.4143153	2.34203425	1.22568849
10	4.4428661	3.2997108	1.4143165	2.58891199	1.35164457

**Table 2 Errors and invariants for a single solitary wave with $c = 1, h = 0.2, k = 0.025,$
 $0 \leq x \leq 100,$ at $t = 10$**

Method	I_1	I_2	I_3	$L_2 \times 10^3$	$L_\infty \times 10^3$
Analytical	4.4428829	3.2998316	1.4142135	0	0
Present	4.4428661	3.2997108	1.4143165	2.58891	1.35164
Pet-Gal.[28]	4.44288	3.29981	1.41416	3.00533	1.68749
Cubic B-splines coll-CN[29]	4.442	3.299	1.413	16.39	9.24
Cubic B-splines coll+PA-CN[29]	4.440	3.296	1.411	20.3	11.2
Cubic B-splines coll[30]	4.44288	3.29983	1.41420	9.30196	5.43718
MQ[35]	4.4428829	3.29978	1.414163	3.914	2.019
IMQ[35]	4.4428611	3.29978	1.414163	3.914	2.019
IQ[35]	4.4428794	3.29978	1.414163	3.914	2.019
GA[35]	4.4428829	3.29978	1.414163	3.914	2.019
TPS[35]	4.4428821	3.29972	1.414104	4.428	2.306

and the computed values of invariants are in good agreement with their analytical values $I_1 = 4.4428829, I_2 = 3.2998316, I_3 = 1.4142135$. Percentage values of the relative error of the conserved quantities I_1, I_2 and I_3 are calculated with respect to the conserved quantities at $t = 0$. Percentage values of relative changes of I_1, I_2 and I_3 are found to be $0.001 \times 10^{-3}\%, 3.389 \times 10^{-3}\%, 7.909 \times 10^{-3}\%$, respectively. Thus, the invariants remain almost constant during the computer run. Table 2 displays a comparison of the values of the invariants and error norms obtained by the present method with those obtained by other methods [28–30, 35]. It can be seen from Table 2 that the error norms obtained by the present method are smaller than other methods [28–30, 35]. Figure 1 shows the motion of a solitary wave with $c = 1, h = 0.2, k = 0.025$ at different time levels. It is observed that the soliton moves to the right at a constant speed and almost unchanged amplitude with increasing time, as expected. At $t = 0$ the amplitude is 1.0 which is located at $x = 40$, while it is 0.999946 which is located at $x = 60$. At times $t = 0$ and $t = 10$, the absolute difference in amplitude is 5×10^{-5} so there is a little change between the amplitudes.

For the second set, the parameters $\mu = 1, c = 0.3, h = 0.1, k = 0.01$ and $x_0 = 40$ with range $[0, 100]$ are chosen to compare the results obtained by the present method with those obtained given in Refs. [28, 30, 32, 34, 35]. So, the solitary wave has amplitude 0.547723 and the computations are done up to time $t = 20$ to obtain the invariants and error norms L_2 and L_∞ at various times. Error norms L_2 and L_∞ and conserved quantities are tabulated in Table 3 together with the results obtained with Refs. [28, 30, 32, 34, 35]. As it is seen from the table, the error norms obtained by the present method are smaller than those



given in Refs. [30, 32] and almost the same as those in Refs. [28, 34, 35]. The agreement between numerical and analytic solutions is perfect which is given by Eq. 23. Percentage values of relative changes of I_1 , I_2 and I_3 are found to be $0.001 \times 10^{-3}\%$, $0.246 \times 10^{-3}\%$, $2.152 \times 10^{-3}\%$, respectively. Moreover, from Table 3, the changing of the invariants I_1 , I_2 and I_3 during the computer run is less than 1×10^{-7} , 3.3×10^{-6} , 3.3×10^{-6} , respectively. The profiles of the solitary wave at different time levels have been shown in Figure 2. The distributions of the errors at time $t = 10$ and $t = 20$ are shown graphically for solitary wave amplitudes 1 and 0.3 in Figure 3. It is seen that the maximum errors are about at the tip of the solitary waves and between -6×10^{-3} and 6×10^{-3} , -2×10^{-4} and 2×10^{-4} , respectively.

3.2 Interaction of two solitary waves

Here the interaction of two solitary waves is studied by using the initial condition given by the linear sum of two well-separated solitary waves having various amplitudes

$$U(x, 0) = \sum_{j=1}^2 A_j \operatorname{sech}(p_j(x - x_j)), \tag{24}$$

where $A_j = \sqrt{c_j}$, $p_j = \sqrt{\frac{c_j}{\mu(c_j+1)}}$, $j = 1, 2$, c_j and x_j are arbitrary constants. The analytical values of the invariants are found by [29]

$$\begin{aligned} I_1 &= \sum_{j=1}^2 \frac{\pi \sqrt{c_j}}{p_j}, \\ I_2 &= \sum_{j=1}^2 \left(\frac{2c_j}{p_j} + \frac{2\mu p_j c_j}{3} \right), \\ I_3 &= \sum_{j=1}^2 \left(\frac{4c_j^2}{3p_j} - \frac{2\mu p_j c_j}{3} \right). \end{aligned} \tag{25}$$

Table 3 Invariants and error norms for a single solitary wave with $c = 0.3, h = 0.1, k = 0.01, 0 \leq x \leq 100$

t	I_1	I_2	I_3	$L_2 \times 10^4$	$L_\infty \times 10^4$
0	3.5820205	1.3450941	0.1537283	0.0000000	0.0000000
2	3.5820205	1.3450944	0.1537280	0.0082694	0.0034843
4	3.5820205	1.3450950	0.1537274	0.0162937	0.0070162
6	3.5820206	1.3450955	0.1537268	0.0242346	0.0105732
8	3.5820206	1.3450960	0.1537264	0.0322064	0.0141521
10	3.5820206	1.3450964	0.1537260	0.0402374	0.0177376
12	3.5820206	1.3450966	0.1537257	0.0483276	0.0213278
14	3.5820206	1.3450969	0.1537255	0.0564695	0.0249138
16	3.5820206	1.3450971	0.1537253	0.0646548	0.0285146
18	3.5820206	1.3450972	0.1537251	0.0728758	0.0321067
20	3.5820204	1.3450974	0.1537250	0.8112594	0.3569076
20[30]	3.58197	1.34508	0.153723	6.06885	2.96650
20[34]	3.581967	1.345076	0.153723	0.508927	0.222284
20[35]MQ	3.5819665	1.3450764	0.153723	0.51498	0.22551
20[35]IMQ	3.5819664	1.3450764	0.153723	0.51498	0.22551
20[35]IQ	3.5819654	1.3450764	0.153723	0.51498	0.22551
20[35]GA	3.5819665	1.3450764	0.153723	0.51498	0.22551
20[35]TPS	3.5819663	1.3450759	0.153723	0.51498	0.26605

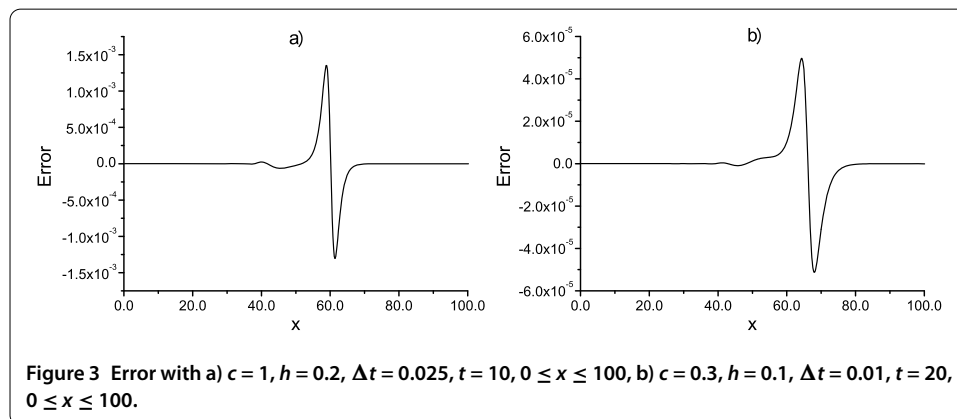
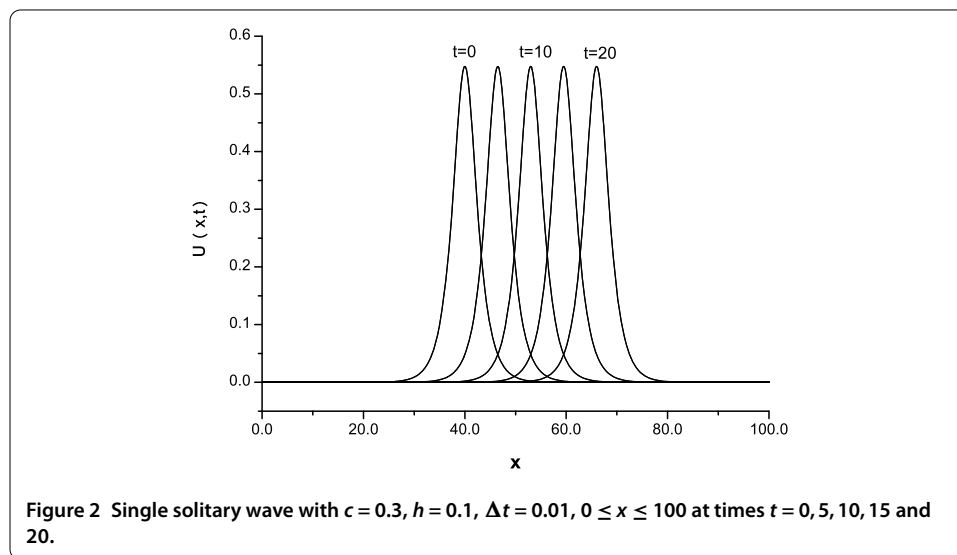


Table 4 Comparison of invariants for the interaction of two solitary waves with results from [34] with $h = 0.2, k = 0.025$ in the region $0 \leq x \leq 250$

Present method				[34]		
t	I_1	I_2	I_3	I_1	I_2	I_3
0	11.4676542	14.6292080	22.8803584	11.467698	14.629277	22.880432
2	11.4678169	14.6282301	22.8813363	11.467698	14.624259	22.860365
4	11.4679819	14.6282293	22.8813371	11.467698	14.619226	22.840279
6	11.4681349	14.6181053	22.8914611	11.467699	14.614169	22.820069
8	11.4675390	14.1393389	23.3702275	11.467700	14.606821	22.787857
10	11.4674118	14.0502062	23.4593602	11.467700	14.603687	22.771773
12	11.4685494	14.6816556	22.8279107	11.467699	14.603056	22.775766
14	11.4687073	14.6648742	22.8446922	11.467699	14.598059	22.756029
16	11.4688627	14.6459207	22.8636457	11.467700	14.593048	22.736127
18	11.4690242	14.6370095	22.8725569	11.467700	14.588061	22.716289
20	11.4691886	14.6331334	22.8764330	11.467701	14.583089	22.696510
20[28]	11.4677	14.6299	22.8806			
20[30]	11.4677	14.6292	22.8809			
20[35]MQ	11.467698	14.583052	22.696539			
20[35]IMQ	11.467679	14.583052	22.696539			
20[35]IQ	11.467690	14.583052	22.696539			
20[35]GA	11.467698	14.583052	22.696539			
20[35]TPS	11.467742	14.582424	22.694269			

For the numerical simulation, the parameters $\mu = 1, h = 0.2, k = 0.025, c_1 = 4, c_2 = 1, x_1 = 25, x_2 = 55$ are used over the range $0 \leq x \leq 250$ to coincide with those used by Refs. [28, 30, 34, 35]. The experiment is run from $t = 0$ to $t = 20$ and the values of invariant quantities I_1, I_2 and I_3 are recorded in Table 4. The analytical values of the invariants for this case are $I_1 = 11.467698, I_2 = 14.629243, I_3 = 22.880466$. A comparison of the values of the invariants obtained by the present method with those obtained in Refs. [28, 30, 34, 35] are listed in Table 4. It is seen that the obtained values of the invariants remain almost constant during the computer run. The development of the interaction of two solitary waves is shown in Figure 4. It can be seen from the figure that at $t = 0$ the wave with larger amplitude is to the left of the second wave with smaller amplitude. Since the taller wave moves faster than the shorter one, it catches up and collides with the shorter one at $t = 8$ and then moves away from the shorter one as time increases. At $t = 20$, the amplitude of larger waves is 2.001090 at the point $x = 127.4$ whereas the amplitude of the smaller one is 0.996399 at the point $x = 92$. It is found that the absolute difference in amplitude is 3.60×10^{-3} for the smaller wave and 1.09×10^{-3} for the larger wave for this algorithm.

3.3 Interaction of three solitary waves

For this problem, the behavior of interaction of three solitary waves having different amplitudes and traveling in the same direction is studied. So, we consider Eq. (2) with the initial condition given by the linear sum of three well-separated solitary waves of different amplitudes

$$U(x, 0) = \sum_{j=1}^3 A_j \operatorname{sech}(p_j(x - x_j)), \tag{26}$$

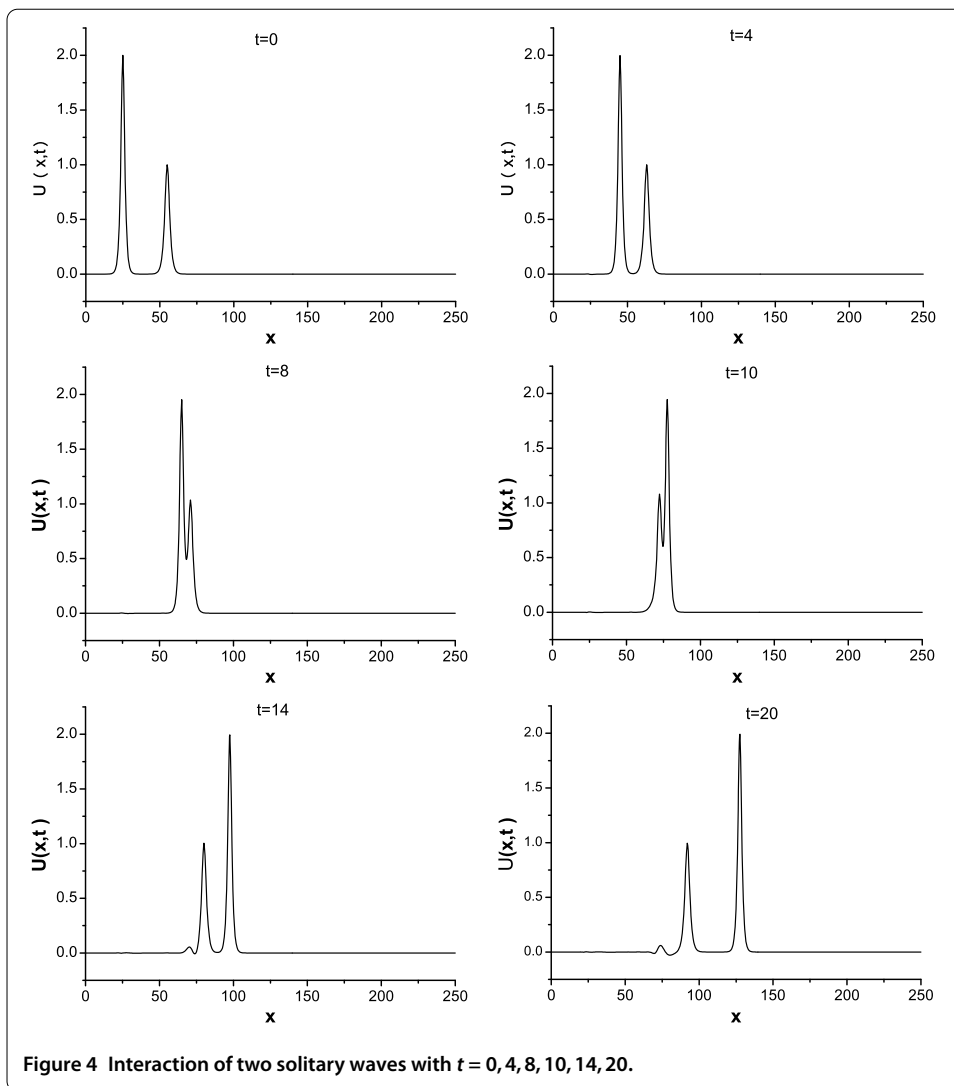


Figure 4 Interaction of two solitary waves with $t = 0, 4, 8, 10, 14, 20$.

where $A_j = \sqrt{c_j}$, $p_j = \sqrt{\frac{c_j}{\mu(c_j+1)}}$, $j = 1, 2, 3$, c_j and x_j are arbitrary constants. The analytical values of the conservation laws are found from Eq. (23) as follows:

$$\begin{aligned}
 I_1 &= \sum_{j=1}^3 \frac{\pi \sqrt{c_j}}{p_j}, \\
 I_2 &= \sum_{j=1}^3 \left(\frac{2c_j}{p_j} + \frac{2\mu p_j c_j}{3} \right), \\
 I_3 &= \sum_{j=1}^3 \left(\frac{4c_j^2}{3p_j} - \frac{2\mu p_j c_j}{3} \right).
 \end{aligned} \tag{27}$$

For the purpose of comparison, parameters $\mu = 1$, $h = 0.2$, $k = 0.025$, $c_1 = 4$, $c_2 = 1$, $c_3 = 0.25$, $x_1 = 15$, $x_2 = 45$, $x_3 = 60$ are used over the region $0 \leq x \leq 250$. During the simulation, time is taken up to $t = 45$. The analytical values of the invariants for this case are $I_1 = 14.9801$, $I_2 = 15.8218$, $I_3 = 22.9923$. A comparison of the values of the invariants

Table 5 Comparison of invariants for the interaction of three solitary waves with results from [34] with $h = 0.2, k = 0.025$ in the region $0 \leq x \leq 250$

Present method				[34]		
t	I_1	I_2	I_3	I_1	I_2	I_3
0	14.9800762	15.8374849	23.0081806	14.980099	15.837528	23.008136
5	14.9381371	15.7382326	23.1074329	14.980105	15.824928	22.957891
10	14.9071292	14.1781087	24.6675567	14.980109	15.807025	22.877972
15	14.8836886	15.3648852	23.4807802	14.980106	15.807032	22.885947
20	14.8503851	15.5659364	23.2797291	14.980106	15.795022	22.837454
25	14.8194163	15.6235556	23.2221098	14.980107	15.782840	22.788852
30	14.7905616	15.5976717	23.2479938	14.980107	15.770634	22.740419
35	14.7636015	15.5610664	23.2845991	14.980108	15.758480	22.692279
40	14.7383184	15.5256320	23.3200335	14.980108	15.746389	22.644448
45	14.7145273	15.4927592	23.3529062	14.968030	15.734374	22.596591
45[30]	13.7043	15.6563	22.9303			
45[35]MQ	14.96814	15.73434	22.596625			
45[35]IMQ	14.96808	15.73434	22.596625			
45[35]IQ	14.96813	15.73434	22.596625			
45[35]GA	14.96810	15.73433	22.596626			
45[35]TPS	14.96824	15.73376	22.594494			

obtained by the present method with those obtained in Refs. [30, 34, 35] are shown in Table 5. It is observed from the table that the obtained values of the invariants remain almost constant during the computer run which are all in good agreement with their analytical values given by Eq. (27). The absolute difference between the values of the conservative constants obtained by the present method at times $t = 0$ and $t = 45$ are $\Delta I_1 = 2.6 \times 10^{-1}$, $\Delta I_2 = 3.4 \times 10^{-1}$, $\Delta I_3 = 3.4 \times 10^{-3}$, respectively. Figure 5 shows the interaction of these solitary waves at different times. As it is seen from the Figure 5, the interaction started at about time $t = 10$, overlapping processes occurred between time $t = 15$ and $t = 40$ and waves started to resume their original shapes after the time $t = 40$.

3.4 The Maxwellian initial condition

Finally, we have studied the development of the Maxwellian initial condition

$$U(x, 0) = \exp(-(x - 40)^2) \tag{28}$$

into a train of solitary waves. As it is known, with the Maxwellian condition (28), the behavior of the solution depends on the values of μ . We study each of the following cases: $\mu = 0.1, \mu = 0.04, \mu = 0.015$ and $\mu = 0.01$. For $\mu = 0.1$, only a single soliton is formed as shown in Figure 6a. When $\mu = 0.04$ and $\mu = 0.015$, two and three stable solitons are formed, respectively, as shown in Figure 6b, c. For $\mu = 0.01$, the Maxwellian initial condition has decayed into four solitary waves as shown in Figure 6d. All figures were drawn up at time $t = 14.5$. The peaks of the well-developed wave lie on a straight line, so that their velocities are linearly dependent on their amplitudes. We also observe a small oscillating tail appearing behind the last wave in all Maxwellian figures. The obtained numerical values of the invariants are given in Table 6.

4 Conclusions

A numerical solution of the MRLW equation based on the quintic B-spline finite element has been successfully presented. The nonlinear term of the equation is linearized by using

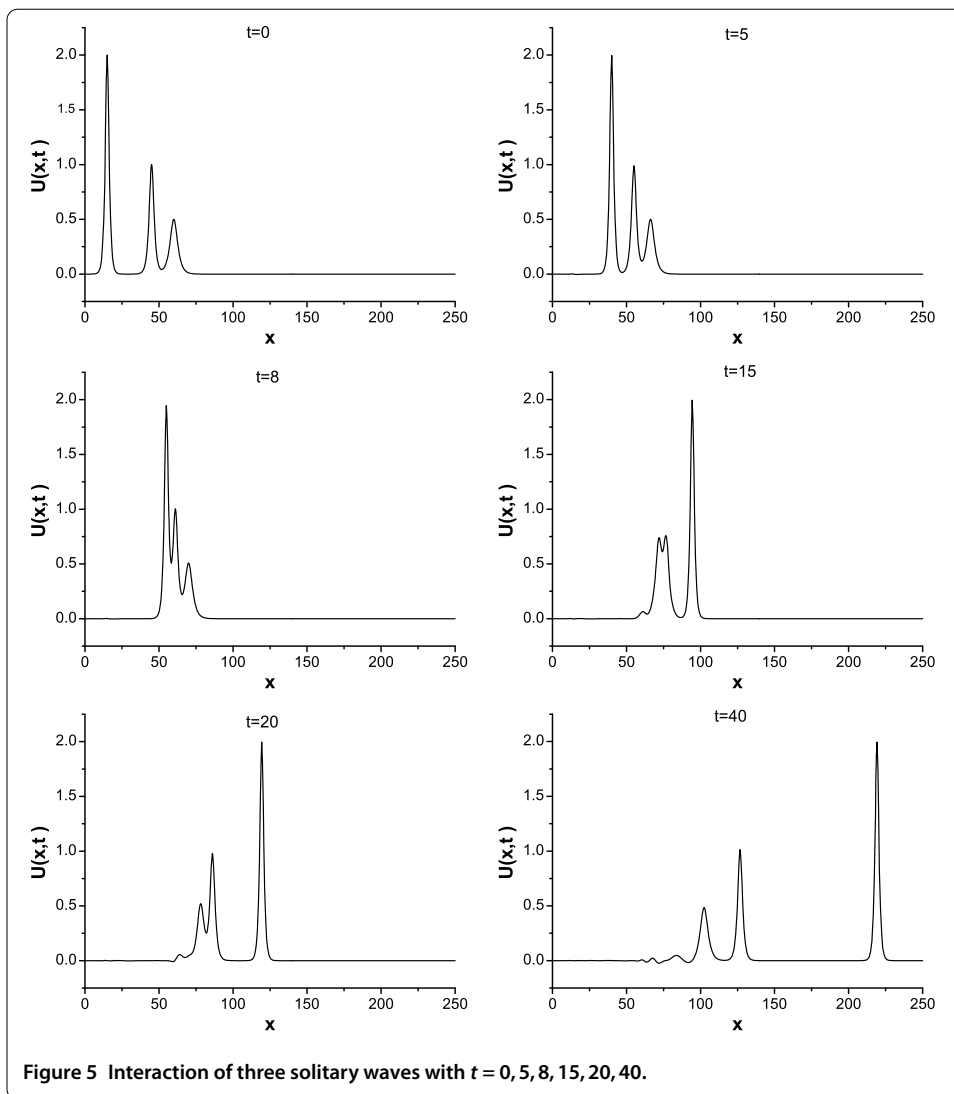
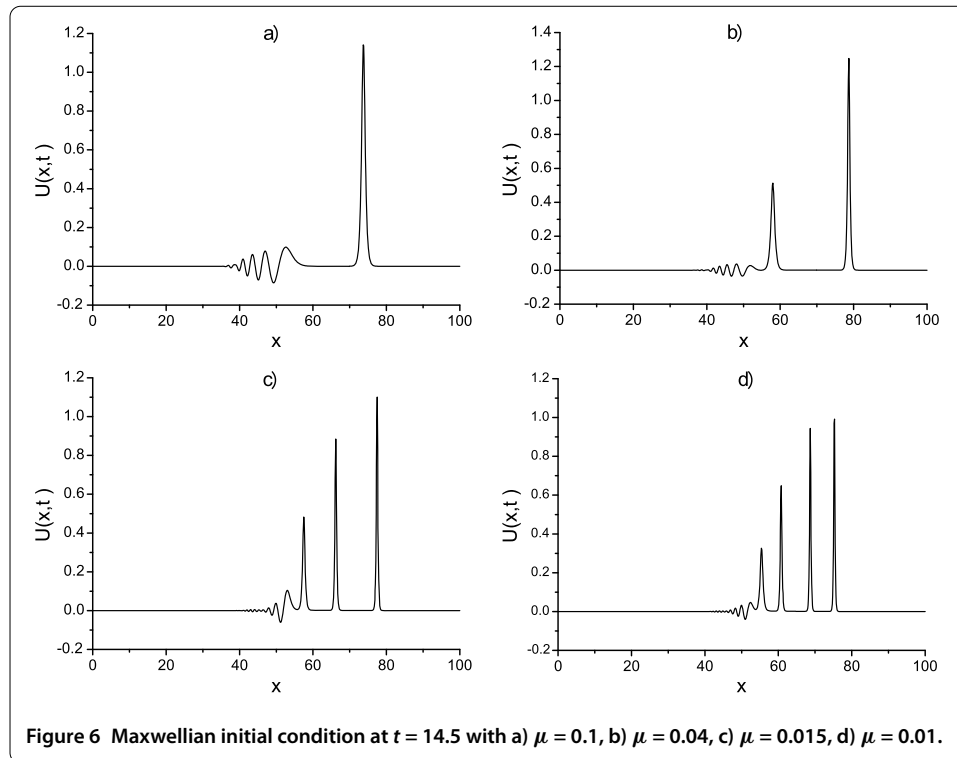


Table 6 Invariants of the MRLW equation using the Maxwellian initial condition

t	μ	I_1	I_2	I_3	μ	I_1	I_2	I_3
0	0.1	1.7724809	1.3786633	0.7609104	0.015	1.7724809	1.2721327	0.8674410
3		1.7721927	1.4728222	0.6667515		1.7436793	1.4180550	0.7215188
6		1.7717480	1.4720618	0.6675119		1.7244620	1.4036726	0.7359011
9		1.7713064	1.4715473	0.6680264		1.7116986	1.3940073	0.7455664
12		1.7708674	1.4711193	0.6684544		1.7021509	1.3870669	0.7525068
15		1.7704309	1.4707290	0.6688447		1.6945087	1.3816736	0.7579001
0	0.01	1.7724809	1.2658662	0.8737075	0.04	1.7724809	1.3034651	0.8361087
3		1.7264258	1.4001430	0.7394307		1.7685556	1.4501960	0.6893777
6		1.7031386	1.3850514	0.7545223		1.7637294	1.4456393	0.6939344
9		1.6885258	1.3755380	0.7640357		1.7592882	1.4415131	0.6980606
12		1.6777156	1.3686563	0.7709174		1.7551780	1.4378815	0.7016922
15		1.6691476	1.3635304	0.7760434		1.7513552	1.4345989	0.7049748

a form given in the paper [36]. Four test problems are studied to examine the performance of the scheme. To show how good and accurate the numerical solutions of the test problems are, the error norms L_2 and L_∞ and the invariant quantities I_1 , I_2 and I_3 have



been used. It is seen that the error norms are sufficiently small and the invariants are well conserved. The method successfully models the motion and interaction of solitary waves. The computed results indicate that the present method is more accurate than some earlier results found in the literature. So, it can be said that the method is a reliable one for obtaining the numerical solutions of a wider range of physically important non-linear partial differential equations.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All the three authors have almost equal contributions to the article. In particular, SBGK participated in the design of the basic outline of the article and equations. NMY participated in the application of the method and obtaining the iterative formulae. YU participated in coding and running the necessary programs. All authors worked together to check and test the programs, to obtain the results, to carry out the literature search. All authors read, checked, corrected and approved the final manuscript.

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Acknowledgements

Dedicated to Professor Hari M Srivastava.

The authors would like to thank the reviewers for their careful reading and making some useful comments which improved the presentation of the paper.

Received: 19 November 2012 Accepted: 29 January 2013 Published: 14 February 2013

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doi:10.1186/1687-2770-2013-27

Cite this article as: Karakoc et al.: Numerical approximation to a solution of the modified regularized long wave equation using quintic B-splines. *Boundary Value Problems* 2013 **2013**:27.