Solvability of a $(k+l)$-order nonlinear difference equation

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Abstract

It is shown that the following $(k+l)$-order nonlinear difference equation

$$x_n = \frac{x_{n-k}x_{n-k-l}}{x_{n-l}\left(a_n + b_n x_{n-k}x_{n-k-l}\right)},\ n \in \mathbb{N}_0,$$

where $k,l \in \mathbb{N}$, $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$ and the initial values x_{-i}, $i=\overline{1,k+l}$, are real numbers, can be solved and extended some results in literature. Also, by using obtained formulas, we give the forbidden set of the initial values for aforementioned equation and study the asymptotic behavior of well-defined solutions of above difference equation for the case $k=3$, $l=k$.

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1 Introduction

Firstly, recall that $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, stand for natural, non-negative integer, integer, real and complex numbers, respectively. If $m,n \in \mathbb{Z}$, $m \le n$ the notation $i=\overline{m,n}$ stands for $\{i \in \mathbb{Z} : m \le i \le n\}$. Big $\mathcal{O}$ notation, called Landau's symbol, is a symbol used in complexity theory, computer science, and mathematics to describe the asymptotic behavior of functions. Landau's symbol comes from the name of a German mathematician Edmund Landau who popularized the notation [1].

Here, we consider the following difference equation

$$x_n = \frac{x_{n-k}x_{n-k-l}}{x_{n-l}\left(a_n + b_n x_{n-k}x_{n-k-l}\right)},\ n \in \mathbb{N}_0, \tag{1}$$

where $k,l \in \mathbb{N}$, $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$ and the initial values x_{-i}, $i=\overline{1,k+l}$, are real numbers.
In [2], among other things, the case $k=1$ and $l=k$ in Eq. (1), which is a natural generalization of the equations given in [3–7],

$$x_{n+1} = \frac{x_n x_{n-k}}{x_{n-k+1}\left(a_n + b_n x_n x_{n-k}\right)},\ n \in \mathbb{N}_0, \tag{2}$$

where $k \in \mathbb{N}$, $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$ and the initial values x_{-i}, $i=\overline{0,k}$, are real numbers, was solved in closed form by a suitable transformation. Similarly, the case $k=2$ and $l=k$ in Eq. (1), which is a natural generalization of the equations given in [8–12],

$$x_n = \frac{x_{n-2}x_{n-k-2}}{x_{n-k}\left(a_n + b_n x_{n-2}x_{n-k-2}\right)},\ n \in \mathbb{N}_0, \tag{3}$$

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where $k \in \mathbb{N}, (a_n)_{n\in\mathbb{N}_0}, (b_n)_{n\in\mathbb{N}_0}$ and the initial values x_{-i}, $i = \overline{1, k+2}$, are real numbers, was solved in closed form and was described long-term behavior of well-defined solutions of Eq. (3) for constant coefficients in [13]. On the other hand, in [14, 15], both the cases $k = 3$, $l = 2$, $a_n = \pm 1$ and $b_n = \pm 1$, for all $n \in \mathbb{N}_0$, and $k = 4$, $l = 1$, $a_n = \pm 1$ and $b_n = \pm 1$, for all $n \in \mathbb{N}_0$, in Eq. (1)

$$x_{n+1} = \frac{x_{n-2}x_{n-4}}{x_{n-1}(\pm 1 \pm x_{n-2}x_{n-4})},\ x_{n+1} = \frac{x_{n-3}x_{n-4}}{x_n(\pm 1 \pm x_{n-3}x_{n-4})},\ n \in \mathbb{N}_0, \tag{4}$$

where the initial conditions are arbitrary non zero real numbers, were studied. Obviously, 4 different difference equations can be written from the first of the equations in (4). Similarly, 4 different difference equations can be written from the second of the equations in (4). Therefore, in total of 8 different difference equations were studied in [14, 15]. The solutions of Eqs. (4) were proved by induction. However, the formulas in [14,15] have not been confirmed by some theoretical explanations.

A natural generalization of two-dimensional system of the Eq. (1) was solved in closed form in [16]. In addition, the asymptotic behavior of well-defined solutions of two-dimensional system of difference equations for the case $k = 2$, $l = k$ was studied in [16]. For related difference equations see also [17–25] and systems of difference equations see also [26–41].

Hence, by the motivation of these results, we consider Eq. (1) which is the most general form of the equations in [2–15], in this paper. Also, we explain form of the solutions of the first equation in Eqs. (4) which are given just by induction method in [14], by using convenient transformation.

Here, our another aim is to solve the Eq. (1) in closed form and to investigate the asymptotic behavior of well-defined solutions of this equation for the case $k = 3$ and $l = k$. Although we studied the asymptotic behavior of well-defined solutions of difference equations system which is a natural generalization of two-dimensional of the Eq. (1) for the case $k = 2$, $l = k$ in [16], we investigate asymptotic behavior of well-defined solutions of the Eq. (1) for $k = 3$, $l = k$ in this paper. Further, the asymptotic behavior of the equation in [14] is also examined in this study.
The paper is organized as follows. In the second section, we will show that Eq. (1) can be solved in closed form. Next, we will describe the forbidden set of the initial values for Eq. (1). In section 3, we will study the solutions of Eq. (1) for $k = 3$, $l = k$ and constant coefficients. Finally, we will investigate asymptotic behavior of solutions.

2 Closed form of equation (1)

In this section, we obtain the solutions of Eq. (1) by using a suitable transformation reducing the equation into a linear type difference equation. If at least one of the initial values x_{-i}, $i = \overline{1, k+l}$, is equal to zero, then the solution of equation (1) is not defined. For example, if $x_{-k-l} = 0$, then $x_0 = 0$ and so x_l is not defined. For $i = \overline{1, k+l-1}$, the other cases are similar. On the other hand, if $x_{n_0} = 0$ $(n_0 \in \mathbb{N}_0)$, $x_n \neq 0$, for $0 \leq n \leq n_0 - 1$, and x_n is defined for $0 \leq n \leq n_0 - 1$, then from Eq. (1) and the selection of number n_0 we have $x_{n_0-k} = 0$ or $x_{n_0-k-l} = 0$, which implies that $n_0 - k < 0$ or $n_0 - k - l < 0$ respectively, that is, $x_{-i} = 0$ for some $i = \overline{1, k+l}$, which is a contradiction. Hence, for every well-defined solutions of Eq. (1), we get $x_{-i} \neq 0$ for every $i = \overline{1, k+l}$, which is equivalent to $x_n \neq 0$ for $n \geq -(k+l)$.
Hence, by employing the substitution $y_n = \frac{1}{x_n x_{n-l}}$, for $n \geq -k$, to Eq. (1), we get the linear k-order

difference equation

$$y_n = a_n y_{n-k} + b_n,\ n \in \mathbb{N}_0. \tag{5}$$

By taking $km+i$, $m \geq -1$ and $i = \overline{0, k-1}$, instead of n in (5), we have the equations

$$y_{km+i} = a_{km+i} y_{k(m-1)+i} + b_{km+i},\ m \in \mathbb{N}_0, \tag{6}$$

which are decompositions of Eq. (5). The solutions of equations in (6) are

$$y_{km+i} = y_{i-k} \prod_{j=0}^{m} a_{kj+i} + \sum_{l=0}^{m} b_{kl+i} \prod_{j=l+1}^{m} a_{kj+i}, \quad m \geq -1, \tag{7}$$

where $i \in \{0, 1, \ldots, k-1\}$. From the substitution $y_n = \frac{1}{x_n x_{n-l}}$, we have

$$x_n = \frac{1}{y_n x_{n-l}} = \frac{y_{n-l}}{y_n} x_{n-2l},\ n \geq l-k, \tag{8}$$

and consequently

$$x_{2lm+i} = x_{i-2l} \prod_{j=0}^{m} \frac{y_{(2j-1)l+i}}{y_{2lj+i}}, \tag{9}$$

for $m \in \mathbb{N}_0$ and $i \in \{l-k, l-k+1, \ldots, 3l-k-1\}$. By the help of the well-known quotient remainder theorem, there exists $k \in \mathbb{N}$ and $s \in \mathbb{N}_0$ such that $n = ks + j_1$ and $j_1 \in \{0, 1, \ldots, k-1\}$. From this and Eq. (9), we have

$$x_{2klm+ks+j_1} = x_{ks+j_1-2kl} \prod_{j=0}^{m} \prod_{n=1}^{k} \frac{y_{2klj+ks+j_1-l(2n-1)}}{y_{2klj+ks+j_1-2l(n-1)}}, \tag{10}$$

for $ks + j_1 \in \{2kl-k-l, 2kl-k-l+1, \ldots, 4kl-k-l-1\}$, if k is odd, and

$$x_{klm+ks+j_1} = x_{ks+j_1-kl} \prod_{j=0}^{m} \prod_{n=1}^{\frac{k}{2}} \frac{y_{klj+ks+j_1-l(2n-1)}}{y_{klj+ks+j_1-2l(n-1)}}, \tag{11}$$

for $ks + j_1 \in \{kl-k-l, kl-k-l+1, \ldots, 2kl-k-l-1\}$, if k is even. Consequently, the solutions of Eq. (1) in closed form are obtained by substituting Eq. (7) in Eq. (10) and Eq. (11).

Now, we give the solutions of Eq. (1) when all the coefficients in Eq. (1) are constant. To do this, we suppose that $a_n = a,\ b_n = b$, for every $n \in \mathbb{N}_0$. Then, Eq. (1) becomes

$$x_n = \frac{x_{n-k} x_{n-k-l}}{x_{n-l}\left(a + b x_{n-k} x_{n-k-l}\right)},\ n \in \mathbb{N}_0. \tag{12}$$

Through this paper we may suppose that $gcd(k,l) = 1$. In fact if $|gcd(k,l)| = f > 1$, which means the greatest common divisor of natural numbers k and l, then $k = fk_1$ and $l = fl_1$ for some $k_1, l_1 \in \mathbb{N}$ such that $gcd(k_1, l_1) = 1$. Similarly, by the help of the well-known quotient remainder theorem, we can write $mf+i$ instead of n in Eq. (12), where $m \in \mathbb{N}_0$ and $i \in \{0,1,\ldots,f-1\}$. Then, it becomes

$$x_{mf+i} = \frac{x_{f(m-k_1)+i}x_{f(m-k_1-l_1)+i}}{x_{f(m-l_1)+i}\left(a+bx_{f(m-k_1)+i}x_{f(m-k_1-l_1)+i}\right)}. \tag{13}$$

By employing the substitution $x_m^{(i)} = x_{mf+i}$, for $m \in \mathbb{N}_0$ and $i \in \{0,1,\ldots,f-1\}$, then Eq. (13) can be written as follows

$$x_m^{(i)} = \frac{x_{m-k_1}^{(i)}x_{m-k_1-l_1}^{(i)}}{x_{m-l_1}^{(i)}\left(a+bx_{m-k_1}^{(i)}x_{m-k_1-l_1}^{(i)}\right)}, \tag{14}$$

which is fundamentally in the form of Eq. (12). Thus, from now on we take the cases when $gcd(k,l)=1$. As we have already mentioned before only the case $k=3,\ l=k$ is suitable for applying Eq. (7) in Eq. (10).

Note that the system given in [16] will reduce to equation (1) in the case $y_n = x_n$, $a_n = \alpha_n$, $b_n = \beta_n$ for all $n \in \mathbb{N}_0$. We can give the following theorem which can be proved as in [16].

Theorem 2.1. Assume that $a_n \neq 0,\ b_n \neq 0$, for $n \in \mathbb{N}_0$. The forbidden set of the initial values for Eq. (1) is given by the set

$$\begin{aligned} \mathcal{F} &= \bigcup_{m\in\mathbb{N}_0}\bigcup_{i=0}^{k-1}\Big\{\left(x_{-(k+l)},\ldots,x_{-1}\right)\in\mathbb{R}^{k+l}:\ x_{i-k}x_{i-k-l} = \frac{1}{c_m}, \text{where} \\ &\qquad c_m := -\sum_{j=0}^{m}\frac{b_{kj+i}}{a_{kj+i}}\prod_{l=0}^{j-1}\frac{1}{a_{kl+i}} \neq 0\Big\} \\ &\qquad \bigcup\bigcup_{j=1}^{k+l}\Big\{\left(x_{-(k+l)},\ldots,x_{-1}\right)\in\mathbb{R}^{k+l} : x_{-j}=0\Big\}. \end{aligned}$$

3 A study of case $k=3, l=k$

In this section, we give the asymptotic behavior of the solutions of Eq. (12) when $k=3$ and $l=k$. In this case, the equation is

$$x_n = \frac{x_{n-3}x_{n-k-3}}{x_{n-k}\left(a+bx_{n-3}x_{n-k-3}\right)},\ n\in\mathbb{N}_0, \tag{15}$$

and the solutions of Eq.(15) can be written from Eq.(10) (in equation (10); $l=3t+r$, where $t\in\mathbb{N}_0,\ r\in\{1,2\}$), as follows:

$$\begin{aligned} x_{6(3t+r)m+3s+j_1} &= x_{3s+j_1-6(3t+r)}\prod_{p=0}^{m}\frac{\left(y_{j_3-3}\left(1-a\right)-b\right)a^{2p(3t+r)+s-t+i_3+1}+b}{\left(y_{j_2-3}\left(1-a\right)-b\right)a^{2p(3t+r)+s+i_2+1}+b} \\ &\times \frac{\left(y_{j_2-3}\left(1-a\right)-b\right)a^{2p(3t+r)+s-3t+i_2-r+1}+b}{\left(y_{j_4-3}\left(1-a\right)-b\right)a^{2p(3t+r)+s-2t+i_4+1}+b} \\ &\times \frac{\left(y_{j_4-3}\left(1-a\right)-b\right)a^{2p(3t+r)+s-5t+i_4-r+1}+b}{\left(y_{j_3-3}\left(1-a\right)-b\right)a^{2p(3t+r)+s-4t+i_3-r+1}+b}, \end{aligned} \tag{16}$$

if $a \neq 1$, and

$$\begin{aligned} x_{6(3t+r)m+3s+j_1} &= x_{3s+j_1-6(3t+r)} \prod_{p=0}^{m} \frac{y_{j_3-3}+b\left(2p\left(3t+r\right)+s-t+i_3+1\right)}{y_{j_2-3}+b\left(2p\left(3t+r\right)+s+i_2+1\right)} \\ &\times \frac{y_{j_2-3}+b\left(2p\left(3t+r\right)+s-3t+i_2-r+1\right)}{y_{j_4-3}+b\left(2p\left(3t+r\right)+s-2t+i_4+1\right)} \\ &\times \frac{y_{j_4-3}+b\left(2p\left(3t+r\right)+s-5t+i_4-r+1\right)}{y_{j_3-3}+b\left(2p\left(3t+r\right)+s-4t+i_3-r+1\right)}, \end{aligned} \tag{17}$$

if $a=1$, where $m \in \mathbb{N}_0$, $r \in \{1,2\}$, $j_1 \in \{3-r, 4-r, 5-r\}$, $3s+j_1 \in \{5k-3, 5k-2, \ldots, 11k-4\}$, $i_2 = \lfloor \frac{j_1}{3} \rfloor$, $i_3 = \lfloor \frac{j_1-r}{3} \rfloor$, $i_4 = \lfloor \frac{j_1-2r}{3} \rfloor$, $j_2 := \begin{cases} 0, & j_1 \equiv 0 \bmod(3) \\ 1, & j_1 \equiv 1 \bmod(3) \\ 2, & j_1 \equiv 2 \bmod(3) \end{cases}$, $j_3 := \begin{cases} 0, & j_1 - r \equiv 0 \bmod(3) \\ 1, & j_1 - r \equiv 1 \bmod(3) \\ 2, & j_1 - r \equiv 2 \bmod(3) \end{cases}$,

$j_4 := \begin{cases} 0, & j_1 - 2r \equiv 0 \bmod(3) \\ 1, & j_1 - 2r \equiv 1 \bmod(3) \\ 2, & j_1 - 3r \equiv 2 \bmod(3) \end{cases}$.

In the following theorem, we obtain the asymptotic behavior of the solutions of Eq. (15) by using the above assumptions in the case when $a \neq -1, b \neq 0$.

Theorem 3.1. Suppose that $a \neq -1$, $b \neq 0$, $k = 3t + r$, $r \in \{1,2\}$ and $t \in \mathbb{N}_0$. Then the following statements hold.

(a) If $|a| > 1$, $y_{-1} \neq \frac{b}{1-a}$, $y_{-2} \neq \frac{b}{1-a}$, $y_{-3} \neq \frac{b}{1-a}$, then $x_m \to 0$, as $m \to \infty$.

(b) If $|a| > 1$, $y_{-1} \neq \frac{b}{1-a}$, $y_{-2} = \frac{b}{1-a}$, $y_{-3} = \frac{b}{1-a}$, then $x_{6(3t+1)m+3s+2} \to 0$, $\left|x_{6(3t+1)m+3s+3}\right| \to \infty$, $x_{6(3t+1)m+3s+4} \to 0$, $\left|x_{6(3t+2)m+3s+1}\right| \to \infty$, $x_{6(3t+2)m+3s+2} \to 0$, $x_{6(3t+2)m+3s+3} \to 0$, $t \in \mathbb{N}_0$ as $m \to \infty$, for every $3s + j_1 \in \{5k-3, 5k-2, \ldots, 11k-4\}$, $j_1 \in \{3-r, 4-r, 5-r\}$, $r \in \{1,2\}$.

(c) If $|a| > 1$, $y_{-1} = \frac{b}{1-a}$, $y_{-2} \neq \frac{b}{1-a}$, $y_{-3} = \frac{b}{1-a}$, then $\left|x_{6(3t+1)m+3s+2}\right| \to \infty$, $x_{6(3t+1)m+3s+3} \to 0$, $x_{6(3t+1)m+3s+4} \to 0$, $x_{6(3t+2)m+3s+1} \to 0$, $x_{6(3t+2)m+3s+2} \to 0$, $\left|x_{6(3t+2)m+3s+3}\right| \to \infty$, $t \in \mathbb{N}_0$ as $m \to \infty$, for every $3s + j_1 \in \{5k-3, 5k-2, \ldots, 11k-4\}$, $j_1 \in \{3-r, 4-r, 5-r\}$, $r \in \{1,2\}$.

(d) If $|a| > 1$, $y_{-1} = \frac{b}{1-a}$, $y_{-2} = \frac{b}{1-a}$, $y_{-3} \neq \frac{b}{1-a}$, then $x_{6(3t+1)m+3s+2} \to 0$, $x_{6(3t+1)m+3s+3} \to 0$, $\left|x_{6(3t+1)m+3s+4}\right| \to \infty$, $x_{6(3t+2)m+3s+1} \to 0$, $\left|x_{6(3t+2)m+3s+2}\right| \to \infty$, $x_{6(3t+2)m+3s+3} \to 0$, $t \in \mathbb{N}_0$ as $m \to \infty$, for every $3s + j_1 \in \{5k-3, 5k-2, \ldots, 11k-4\}$, $j_1 \in \{3-r, 4-r, 5-r\}$, $r \in \{1,2\}$.

(e) If $|a| > 1$, $y_{-1} = \frac{b}{1-a}$, $y_{-2} \neq \frac{b}{1-a}$, $y_{-3} \neq \frac{b}{1-a}$, then $x_{6(3t+1)m+3s+2}$, $x_{6(3t+1)m+3s+4}$, $x_{6(3t+2)m+3s+2}$, $x_{6(3t+2)m+3s+3}$ are constant and $x_{6(3t+1)m+3s+3} \to 0$, $x_{6(3t+2)m+3s+1} \to 0$, $t \in \mathbb{N}_0$ as $m \to \infty$, for every $3s + j_1 \in \{5k-3, 5k-2, \ldots, 11k-4\}$, $j_1 \in \{3-r, 4-r, 5-r\}$, $r \in \{1,2\}$.

(f) If $|a|>1$, $y_{-1}\neq\frac{b}{1-a}$, $y_{-2}=\frac{b}{1-a}$, $y_{-3}\neq\frac{b}{1-a}$, then $x_{6(3t+1)m+3s+3}$, $x_{6(3t+1)m+3s+4}$, $x_{6(3t+2)m+3s+1}$, $x_{6(3t+2)m+3s+2}$ are constant and $x_{6(3t+1)m+3s+2}\to 0$, $x_{6(3t+2)m+3s+3}\to 0$, $t\in\mathbb{N}_0$ as $m\to\infty$, for every $3s+j_1\in\{5k-3,5k-2,\ldots,11k-4\}$, $j_1\in\{3-r,4-r,5-r\}$, $r\in\{1,2\}$.

(g) If $|a|>1$, $y_{-1}\neq\frac{b}{1-a}$, $y_{-2}\neq\frac{b}{1-a}$, $y_{-3}=\frac{b}{1-a}$, then $x_{6(3t+1)m+3s+2}$, $x_{6(3t+1)m+3s+3}$, $x_{6(3t+2)m+3s+1}$, $x_{6(3t+2)m+3s+3}$ are constant and $x_{6(3t+1)m+3s+4}\to 0$, $x_{6(3t+2)m+3s+2}\to 0$, $t\in\mathbb{N}_0$ as $m\to\infty$, for every $3s+j_1\in\{5k-3,5k-2,\ldots,11k-4\}$, $j_1\in\{3-r,4-r,5-r\}$, $r\in\{1,2\}$.

(h) If $|a|<1$, then the sequences $(x_{6km+i})_{m\in\mathbb{N}_0}$, $\forall i\in\{0,1,\ldots,6k-1\}$, are convergent.

(i) If $y_{-1}=y_{-2}=y_{-3}=\frac{b}{1-a}$, then $x_{6km+j}=x_{j-6k}$, for $m\in\mathbb{N}_0$, $j\in\{5k-3,5k-2,\ldots,11k-4\}$.

(j) If $a=0$, then the sequence $(x_n)_{n\geq -k-3}$ is $2k$ periodic.

(k) If $a=1$, then $x_m\to 0$, as $m\to\infty$.

Proof. Suppose that

$$w_{m,1}^{3s+2} = \frac{(y_{-2}(1-a)-b)a^{2m(3t+1)+s-t+1}+b}{(y_{-1}(1-a)-b)a^{2m(3t+1)+s+1}+b}\,\frac{(y_{-1}(1-a)-b)a^{2m(3t+1)+s-3t}+b}{(y_{-3}(1-a)-b)a^{2m(3t+1)+s-2t+1}+b} \times \frac{(y_{-3}(1-a)-b)a^{2m(3t+1)+s-5t}+b}{(y_{-2}(1-a)-b)a^{2m(3t+1)+s-4t}+b}, \tag{18}$$

for $3s+2\in\{5k-3,5k-2,\ldots,11k-4\}$,

$$w_{m,1}^{3s+3} = \frac{(y_{-1}(1-a)-b)a^{2m(3t+1)+s-t+1}+b}{(y_{-3}(1-a)-b)a^{2m(3t+1)+s+2}+b}\,\frac{(y_{-3}(1-a)-b)a^{2m(3t+1)+s-3t+1}+b}{(y_{-2}(1-a)-b)a^{2m(3t+1)+s-2t+1}+b} \times \frac{(y_{-2}(1-a)-b)a^{2m(3t+1)+s-5t}+b}{(y_{-1}(1-a)-b)a^{2m(3t+1)+s-4t}+b}, \tag{19}$$

for $3s+3\in\{5k-3,5k-2,\ldots,11k-4\}$,

$$w_{m,1}^{3s+4} = \frac{(y_{-3}(1-a)-b)a^{2m(3t+1)+s-t+2}+b}{(y_{-2}(1-a)-b)a^{2m(3t+1)+s+2}+b}\,\frac{(y_{-2}(1-a)-b)a^{2m(3t+1)+s-3t+1}+b}{(y_{-1}(1-a)-b)a^{2m(3t+1)+s-2t+1}+b} \times \frac{(y_{-1}(1-a)-b)a^{2m(3t+1)+s-5t}+b}{(y_{-3}(1-a)-b)a^{2m(3t+1)+s-4t+1}+b}, \tag{20}$$

for $3s+4\in\{5k-3,5k-2,\ldots,11k-4\}$,

$$w_{m,2}^{3s+1} = \frac{(y_{-1}(1-a)-b)a^{2m(3t+2)+s-t}+b}{(y_{-2}(1-a)-b)a^{2m(3t+2)+s+1}+b}\,\frac{(y_{-2}(1-a)-b)a^{2m(3t+2)+s-3t-1}+b}{(y_{-3}(1-a)-b)a^{2m(3t+2)+s-2t}+b} \times \frac{(y_{-3}(1-a)-b)a^{2m(3t+2)+s-5t-2}+b}{(y_{-1}(1-a)-b)a^{2m(3t+2)+s-4t-2}+b}, \tag{21}$$

for $3s+1 \in \{5k-3, 5k-2, \ldots, 11k-4\}$,

$$\begin{aligned} w_{m,2}^{3s+2} &= \frac{(y_{-3}(1-a)-b)a^{2m(3t+2)+s-t+1}+b}{(y_{-1}(1-a)-b)a^{2m(3t+2)+s+1}+b}\frac{(y_{-1}(1-a)-b)a^{2m(3t+2)+s-3t-1}+b}{(y_{-2}(1-a)-b)a^{2m(3t+2)+s-2t}+b} \\ &\times \frac{(y_{-2}(1-a)-b)a^{2m(3t+2)+s-5t-2}+b}{(y_{-3}(1-a)-b)a^{2m(3t+2)+s-4t-1}+b}, \end{aligned} \tag{22}$$

for $3s+2 \in \{5k-3, 5k-2, \ldots, 11k-4\}$ and

$$\begin{aligned} w_{m,2}^{3s+3} &= \frac{(y_{-2}(1-a)-b)a^{2m(3t+2)+s-t+1}+b}{(y_{-3}(1-a)-b)a^{2m(3t+2)+s+2}+b}\frac{(y_{-3}(1-a)-b)a^{2m(3t+2)+s-3t}+b}{(y_{-1}(1-a)-b)a^{2m(3t+2)+s-2t}+b} \\ &\times \frac{(y_{-1}(1-a)-b)a^{2m(3t+2)+s-5t-2}+b}{(y_{-2}(1-a)-b)a^{2m(3t+2)+s-4t-1}+b}, \end{aligned} \tag{23}$$

for $3s+3 \in \{5k-3, 5k-2, \ldots, 11k-4\}$.
(a): Assume that $y_{-1} \neq \frac{b}{1-a}$, $y_{-2} \neq \frac{b}{1-a}$, $y_{-3} \neq \frac{b}{1-a}$. From (18)-(23), we have

$$\lim_{m\to\infty} w_{m,1}^{3s+2} = \lim_{m\to\infty} w_{m,1}^{3s+3} = \lim_{m\to\infty} w_{m,1}^{3s+4} = \frac{1}{a^{3t+1}},$$

$$\lim_{m\to\infty} w_{m,2}^{3s+1} = \lim_{m\to\infty} w_{m,2}^{3s+2} = \lim_{m\to\infty} w_{m,2}^{3s+3} = \frac{1}{a^{3t+2}}.$$

The results follow from the condition $|a| > 1$.
(b): From the condition $y_{-1} \neq \frac{b}{1-a}$, $y_{-2} = \frac{b}{1-a}$, $y_{-3} = \frac{b}{1-a}$, from (18)-(23), we have

$$\lim_{m\to\infty} w_{m,1}^{3s+2} = \frac{1}{a^{3t+1}},\ \lim_{m\to\infty} w_{m,1}^{3s+3} = a^{3t+1},\ \lim_{m\to\infty} w_{m,1}^{3s+4} = \frac{1}{a^{3t+1}},$$

$$\lim_{m\to\infty} w_{m,2}^{3s+1} = a^{3t+2},\ \lim_{m\to\infty} w_{m,2}^{3s+2} = \frac{1}{a^{3t+2}},\ \lim_{m\to\infty} w_{m,2}^{3s+3} = \frac{1}{a^{3t+2}}.$$

The results follow from the condition $|a| > 1$.
Proofs of the (c)-(d) are not given here since they could be obtained similar with proof of the (a)-(b).
(e): Suppose that $y_{-1} = \frac{b}{1-a}$, $y_{-2} \neq \frac{b}{1-a}$, $y_{-3} \neq \frac{b}{1-a}$, from (18)-(23), we have

$$\lim_{m\to\infty} w_{m,1}^{3s+2} = 1,\ \lim_{m\to\infty} w_{m,1}^{3s+3} = \frac{1}{a^{2(3t+1)}},\ \lim_{m\to\infty} w_{m,1}^{3s+4} = 1,$$

$$\lim_{m\to\infty} w_{m,2}^{3s+1} = \frac{1}{a^{2(3t+2)}},\ \lim_{m\to\infty} w_{m,2}^{3s+2} = 1,\ \lim_{m\to\infty} w_{m,2}^{3s+3} = 1.$$

The results follow from the condition $|a| > 1$.
Proofs of the (f)-(g) are not given here since they can be obtained similar with proof of the (e).

(h): Employing the Taylor expansion for $(1+x)^{-1}$ on the interval $(-\epsilon,\epsilon)$, where $\epsilon>0$, we get, for sufficiently large m,

$$\begin{aligned}
w_{m,1}^{3s+2} &= \frac{(y_{-2}(1-a)-b)a^{2m(3t+1)+s-t+1}+b}{(y_{-1}(1-a)-b)a^{2m(3t+1)+s+1}+b}\frac{(y_{-1}(1-a)-b)a^{2m(3t+1)+s-3t}+b}{(y_{-3}(1-a)-b)a^{2m(3t+1)+s-2t+1}+b}\\
&\times \frac{(y_{-3}(1-a)-b)a^{2m(3t+1)+s-5t}+b}{(y_{-2}(1-a)-b)a^{2m(3t+1)+s-4t}+b}\\
&= 1+\frac{(y_{-1}(1-a)-b)}{b}\left(\frac{1}{a^{3t+1}}-1\right)a^{2m(3t+1)+s+1}\\
&+ \frac{(y_{-2}(1-a)-b)}{b}\left(\frac{1}{a^{t}}-\frac{1}{a^{4t+1}}\right)a^{2m(3t+1)+s+1}\\
&+ \frac{(y_{-3}(1-a)-b)}{b}\left(\frac{1}{a^{5t+1}}-\frac{1}{a^{2t}}\right)a^{2m(3t+1)+s+1}\\
&+ \frac{(y_{-1}(1-a)-b)}{b}\left(\frac{1}{a^{3t+1}}-1\right)\frac{(y_{-2}(1-a)-b)}{b}\left(\frac{1}{a^{t}}-\frac{1}{a^{4t+1}}\right)a^{4m(3t+1)+2s+2}\\
&+ \frac{(y_{-1}(1-a)-b)}{b}\left(\frac{1}{a^{3t+1}}-1\right)\frac{(y_{-3}(1-a)-b)}{b}\left(\frac{1}{a^{5t+1}}-\frac{1}{a^{2t}}\right)a^{4m(3t+1)+2s+2}\\
&+ \frac{(y_{-2}(1-a)-b)}{b}\left(\frac{1}{a^{t}}-\frac{1}{a^{4t+1}}\right)\frac{(y_{-3}(1-a)-b)}{b}\left(\frac{1}{a^{5t+1}}-\frac{1}{a^{2t}}\right)a^{4m(3t+1)+2s+2}\\
&+ \mathcal{O}\left(a^{4m}\right) \qquad (24)
\end{aligned}$$

if $3s+2\in\{5k-3,5k-2,\dots,11k-4\}$,

$$\begin{aligned}
w_{m,1}^{3s+3} &= \frac{(y_{-1}(1-a)-b)a^{2m(3t+1)+s-t+1}+b}{(y_{-3}(1-a)-b)a^{2m(3t+1)+s+2}+b}\frac{(y_{-3}(1-a)-b)a^{2m(3t+1)+s-3t+1}+b}{(y_{-2}(1-a)-b)a^{2m(3t+1)+s-2t+1}+b}\\
&\times \frac{(y_{-2}(1-a)-b)a^{2m(3t+1)+s-5t}+b}{(y_{-1}(1-a)-b)a^{2m(3t+1)+s-4t}+b}\\
&= 1+\frac{(y_{-1}(1-a)-b)}{b}\left(\frac{1}{a^{t}}-\frac{1}{a^{4t+1}}\right)a^{2m(3t+1)+s+1}\\
&+ \frac{(y_{-2}(1-a)-b)}{b}\left(\frac{1}{a^{5t+1}}-\frac{1}{a^{2t}}\right)a^{2m(3t+1)+s+1}\\
&+ \frac{(y_{-3}(1-a)-b)}{b}\left(\frac{1}{a^{3t}}-a\right)a^{2m(3t+1)+s+1}\\
&+ \frac{(y_{-1}(1-a)-b)}{b}\left(\frac{1}{a^{t}}-\frac{1}{a^{4t+1}}\right)\frac{(y_{-2}(1-a)-b)}{b}\left(\frac{1}{a^{5t+1}}-\frac{1}{a^{2t}}\right)a^{4m(3t+1)+2s+2}\\
&+ \frac{(y_{-1}(1-a)-b)}{b}\left(\frac{1}{a^{t}}-\frac{1}{a^{4t+1}}\right)\frac{(y_{-3}(1-a)-b)}{b}\left(\frac{1}{a^{3t}}-a\right)a^{4m(3t+1)+2s+2}\\
&+ \frac{(y_{-2}(1-a)-b)}{b}\left(\frac{1}{a^{5t+1}}-\frac{1}{a^{2t}}\right)\frac{(y_{-3}(1-a)-b)}{b}\left(\frac{1}{a^{3t}}-a\right)a^{4m(3t+1)+2s+2}\\
&+ \mathcal{O}\left(a^{4m}\right), \qquad (25)
\end{aligned}$$

if $3s+3 \in \{5k-3, 5k-2, \dots, 11k-4\}$,

$$\begin{aligned}
w_{m,1}^{3s+4} &= \frac{(y_{-3}(1-a)-b)a^{2m(3t+1)+s-t+2}+b}{(y_{-2}(1-a)-b)a^{2m(3t+1)+s+2}+b}\frac{(y_{-2}(1-a)-b)a^{2m(3t+1)+s-3t+1}+b}{(y_{-1}(1-a)-b)a^{2m(3t+1)+s-2t+1}+b}\\
&\times \frac{(y_{-1}(1-a)-b)a^{2m(3t+1)+s-5t}+b}{(y_{-3}(1-a)-b)a^{2m(3t+1)+s-4t+1}+b}\\
&= 1+\frac{(y_{-1}(1-a)-b)}{b}\left(\frac{1}{a^{5t+1}}-\frac{1}{a^{2t}}\right)a^{2m(3t+1)+s+1}\\
&+ \frac{(y_{-2}(1-a)-b)}{b}\left(\frac{1}{a^{3t}}-a\right)a^{2m(3t+1)+s+1}\\
&+ \frac{(y_{-3}(1-a)-b)}{b}\left(\frac{1}{a^{t-1}}-\frac{1}{a^{4t}}\right)a^{2m(3t+1)+s+1}\\
&+ \frac{(y_{-1}(1-a)-b)}{b}\left(\frac{1}{a^{5t+1}}-\frac{1}{a^{2t}}\right)\frac{(y_{-2}(1-a)-b)}{b}\left(\frac{1}{a^{3t}}-a\right)a^{4m(3t+1)+2s+2}\\
&+ \frac{(y_{-1}(1-a)-b)}{b}\left(\frac{1}{a^{5t+1}}-\frac{1}{a^{2t}}\right)\frac{(y_{-3}(1-a)-b)}{b}\left(\frac{1}{a^{t-1}}-\frac{1}{a^{4t}}\right)a^{4m(3t+1)+2s+2}\\
&+ \frac{(y_{-2}(1-a)-b)}{b}\left(\frac{1}{a^{3t}}-a\right)\frac{(y_{-3}(1-a)-b)}{b}\left(\frac{1}{a^{t-1}}-\frac{1}{a^{4t}}\right)a^{4m(3t+1)+2s+2}\\
&+ \mathcal{O}\left(a^{4m}\right), \qquad (26)
\end{aligned}$$

if $3s+4 \in \{5k-3, 5k-2, \dots, 11k-4\}$,

$$\begin{aligned}
w_{m,2}^{3s+1} &= \frac{(y_{-1}(1-a)-b)a^{2m(3t+2)+s-t}+b}{(y_{-2}(1-a)-b)a^{2m(3t+2)+s+1}+b}\frac{(y_{-2}(1-a)-b)a^{2m(3t+2)+s-3t-1}+b}{(y_{-3}(1-a)-b)a^{2m(3t+2)+s-2t}+b}\\
&\times \frac{(y_{-3}(1-a)-b)a^{2m(3t+2)+s-5t-2}+b}{(y_{-1}(1-a)-b)a^{2m(3t+2)+s-4t-2}+b}\\
&= 1+\frac{(y_{-1}(1-a)-b)}{b}\left(\frac{1}{a^{t+1}}-\frac{1}{a^{4t+3}}\right)a^{2m(3t+2)+s+1}\\
&+ \frac{(y_{-2}(1-a)-b)}{b}\left(\frac{1}{a^{3t+2}}-1\right)a^{2m(3t+2)+s+1}\\
&+ \frac{(y_{-3}(1-a)-b)}{b}\left(\frac{1}{a^{5t+3}}-\frac{1}{a^{2t+1}}\right)a^{2m(3t+2)+s+1}\\
&+ \frac{(y_{-1}(1-a)-b)}{b}\left(\frac{1}{a^{t+1}}-\frac{1}{a^{4t+3}}\right)\frac{(y_{-2}(1-a)-b)}{b}\left(\frac{1}{a^{3t+2}}-1\right)a^{4m(3t+2)+2s+2}\\
&+ \frac{(y_{-1}(1-a)-b)}{b}\left(\frac{1}{a^{t+1}}-\frac{1}{a^{4t+3}}\right)\frac{(y_{-3}(1-a)-b)}{b}\left(\frac{1}{a^{5t+3}}-\frac{1}{a^{2t+1}}\right)a^{4m(3t+2)+2s+2}\\
&+ \frac{(y_{-2}(1-a)-b)}{b}\left(\frac{1}{a^{3t+2}}-1\right)\frac{(y_{-3}(1-a)-b)}{b}\left(\frac{1}{a^{5t+3}}-\frac{1}{a^{2t+1}}\right)a^{4m(3t+2)+2s+2}\\
&+ \mathcal{O}\left(a^{4m}\right), \qquad (27)
\end{aligned}$$

if $3s+1 \in \{5k-3, 5k-2, \dots, 11k-4\}$,

$$\begin{aligned}
w_{m,2}^{3s+2} &= \frac{(y_{-3}(1-a)-b)a^{2m(3t+2)+s-t+1}+b}{(y_{-1}(1-a)-b)a^{2m(3t+2)+s+1}+b}\frac{(y_{-1}(1-a)-b)a^{2m(3t+2)+s-3t-1}+b}{(y_{-2}(1-a)-b)a^{2m(3t+2)+s-2t}+b}\\
&\times \frac{(y_{-2}(1-a)-b)a^{2m(3t+2)+s-5t-2}+b}{(y_{-3}(1-a)-b)a^{2m(3t+2)+s-4t-1}+b}\\
&= 1+\frac{(y_{-1}(1-a)-b)}{b}\left(\frac{1}{a^{3t+2}}-1\right)a^{2m(3t+2)+s+1}\\
&+ \frac{(y_{-2}(1-a)-b)}{b}\left(\frac{1}{a^{5t+3}}-\frac{1}{a^{2t+1}}\right)a^{2m(3t+2)+s+1}\\
&+ \frac{(y_{-3}(1-a)-b)}{b}\left(\frac{1}{a^{t}}-\frac{1}{a^{4t+2}}\right)a^{2m(3t+2)+s+1}\\
&+ \frac{(y_{-1}(1-a)-b)}{b}\left(\frac{1}{a^{3t+2}}-1\right)\frac{(y_{-2}(1-a)-b)}{b}\left(\frac{1}{a^{5t+3}}-\frac{1}{a^{2t+1}}\right)a^{4m(3t+2)+2s+2}\\
&+ \frac{(y_{-1}(1-a)-b)}{b}\left(\frac{1}{a^{3t+2}}-1\right)\frac{(y_{-3}(1-a)-b)}{b}\left(\frac{1}{a^{t}}-\frac{1}{a^{4t+2}}\right)a^{4m(3t+2)+2s+2}\\
&+ \frac{(y_{-2}(1-a)-b)}{b}\left(\frac{1}{a^{5t+3}}-\frac{1}{a^{2t+1}}\right)\frac{(y_{-3}(1-a)-b)}{b}\left(\frac{1}{a^{t}}-\frac{1}{a^{4t+2}}\right)a^{4m(3t+2)+2s+2}\\
&+ \mathcal{O}\left(a^{4m}\right), \qquad (28)
\end{aligned}$$

if $3s+2 \in \{5k-3, 5k-2, \dots, 11k-4\}$ and

$$\begin{aligned}
w_{m,2}^{3s+3} &= \frac{(y_{-2}(1-a)-b)a^{2m(3t+2)+s-t+1}+b}{(y_{-3}(1-a)-b)a^{2m(3t+2)+s+2}+b}\frac{(y_{-3}(1-a)-b)a^{2m(3t+2)+s-3t}+b}{(y_{-1}(1-a)-b)a^{2m(3t+2)+s-2t}+b}\\
&\times \frac{(y_{-1}(1-a)-b)a^{2m(3t+2)+s-5t-2}+b}{(y_{-2}(1-a)-b)a^{2m(3t+2)+s-4t-1}+b}\\
&= 1+\frac{(y_{-1}(1-a)-b)}{b}\left(\frac{1}{a^{5t+3}}-\frac{1}{a^{2t+1}}\right)a^{2m(3t+2)+s+1}\\
&+ \frac{(y_{-2}(1-a)-b)}{b}\left(\frac{1}{a^{t}}-\frac{1}{a^{4t+2}}\right)a^{2m(3t+2)+s+1}\\
&+ \frac{(y_{-3}(1-a)-b)}{b}\left(\frac{1}{a^{3t+1}}-a\right)a^{2m(3t+2)+s+1}\\
&+ \frac{(y_{-1}(1-a)-b)}{b}\left(\frac{1}{a^{5t+3}}-\frac{1}{a^{2t+1}}\right)\frac{(y_{-2}(1-a)-b)}{b}\left(\frac{1}{a^{t}}-\frac{1}{a^{4t+2}}\right)a^{4m(3t+2)+2s+2}\\
&+ \frac{(y_{-1}(1-a)-b)}{b}\left(\frac{1}{a^{5t+3}}-\frac{1}{a^{2t+1}}\right)\frac{(y_{-3}(1-a)-b)}{b}\left(\frac{1}{a^{3t+1}}-a\right)a^{4m(3t+2)+2s+2}\\
&+ \frac{(y_{-2}(1-a)-b)}{b}\left(\frac{1}{a^{t}}-\frac{1}{a^{4t+2}}\right)\frac{(y_{-3}(1-a)-b)}{b}\left(\frac{1}{a^{3t+1}}-a\right)a^{4m(3t+2)+2s+2}\\
&+ \mathcal{O}\left(a^{4m}\right), \qquad (29)
\end{aligned}$$

if $3s+3 \in \{5k-3, 5k-2, \dots, 11k-4\}$. The results follow from (24)-(29) and the condition $|a| < 1$.

(i): This result can be seen easily from the assumption $y_{-1} = y_{-2} = y_{-3} = \frac{b}{1-a}$ and some simple calculation.
(j): If $a = 0$. From Eq. (15), we get

$$x_n = \frac{1}{bx_{n-k}} = \frac{1}{b\frac{1}{bx_{n-2k}}} = x_{n-2k}, \tag{30}$$

From Eq. (30), the result can be seen easily.
(k): Firstly, before giving the proof of (k) we introduce the following notation

$$\begin{aligned} \widehat{w}_{m,1}^{3s+2} &= \frac{y_{-2} + b\,(2m(3t+1)+s-t+1)}{y_{-1} + b\,(2m(3t+1)+s+1)} \frac{y_{-1} + b\,(2m(3t+1)+s-3t)}{y_{-3} + b\,(2m(3t+1)+s-2t+1)} \\ &\times \frac{y_{-3} + b\,(2m(3t+1)+s-5t)}{y_{-2} + b\,(2m(3t+1)+s-4t)}, \end{aligned} \tag{31}$$

for $3s+2 \in \{5k-3, 5k-2, \dots, 11k-4\}$,

$$\begin{aligned} \widehat{w}_{m,1}^{3s+3} &= \frac{y_{-1} + b\,(2m(3t+1)+s-t+1)}{y_{-3} + b\,(2m(3t+1)+s+2)} \frac{y_{-3} + b\,(2m(3t+1)+s-3t+1)}{y_{-2} + b\,(2m(3t+1)+s-2t+1)} \\ &\times \frac{y_{-2} + b\,(2m(3t+1)+s-5t)}{y_{-1} + b\,(2m(3t+1)+s-4t)}, \end{aligned} \tag{32}$$

for $3s+3 \in \{5k-3, 5k-2, \dots, 11k-4\}$,

$$\begin{aligned} \widehat{w}_{m,1}^{3s+4} &= \frac{y_{-3} + b\,(2m(3t+1)+s-t+2)}{y_{-2} + b\,(2m(3t+1)+s+2)} \frac{y_{-2} + b\,(2m(3t+1)+s-3t+1)}{y_{-1} + b\,(2m(3t+1)+s-2t+1)} \\ &\times \frac{y_{-1} + b\,(2m(3t+1)+s-5t)}{y_{-3} + b\,(2m(3t+1)+s-4t+1)}, \end{aligned} \tag{33}$$

for $3s+4 \in \{5k-3, 5k-2, \dots, 11k-4\}$,

$$\begin{aligned} \widehat{w}_{m,2}^{3s+1} &= \frac{y_{-1} + b\,(2m(3t+2)+s-t)}{y_{-2} + b\,(2m(3t+2)+s+1)} \frac{y_{-2} + b\,(2m(3t+2)+s-3t-1)}{y_{-3} + b\,(2m(3t+2)+s-2t)} \\ &\times \frac{y_{-3} + b\,(2m(3t+2)+s-5t-2)}{y_{-1} + b\,(2m(3t+2)+s-4t-2)}, \end{aligned} \tag{34}$$

for $3s+1 \in \{5k-3, 5k-2, \dots, 11k-4\}$,

$$\begin{aligned} \widehat{w}_{m,2}^{3s+2} &= \frac{y_{-3} + b\,(2m(3t+2)+s-t+1)}{y_{-1} + b\,(2m(3t+2)+s+1)} \frac{y_{-1} + b\,(2m(3t+2)+s-3t-1)}{y_{-2} + b\,(2m(3t+2)+s-2t)} \\ &\times \frac{y_{-2} + b\,(2m(3t+2)+s-5t-2)}{y_{-3} + b\,(2m(3t+2)+s-4t-1)}, \end{aligned} \tag{35}$$

for $3s+2 \in \{5k-3, 5k-2, \dots, 11k-4\}$,

$$\begin{aligned} \widehat{w}_{m,2}^{3s+3} &= \frac{y_{-2} + b\,(2m(3t+2)+s-t+1)}{y_{-3} + b\,(2m(3t+2)+s+2)} \frac{y_{-3} + b\,(2m(3t+2)+s-3t)}{y_{-1} + b\,(2m(3t+2)+s-2t)} \\ &\times \frac{y_{-1} + b\,(2m(3t+2)+s-5t-2)}{y_{-2} + b\,(2m(3t+2)+s-4t-1)}, \end{aligned} \tag{36}$$

for $3s+3 \in \{5k-3, 5k-2, \dots, 11k-4\}$.
Suppose that $a=1$.

$$\begin{aligned}
\widehat{w}_{m,1}^{3s+2} &= \left(1+\frac{-b(3t+1)}{2bm(3t+1)+b(s+1)+y_{-1}}\right)\left(1+\frac{b(3t+1)}{2bm(3t+1)+b(s-4t)+y_{-2}}\right)\\
&\times \left(1+\frac{-b(3t+1)}{2bm(3t+1)+b(s-2t+1)+y_{-3}}\right)\\
&= \left(1-\frac{1}{2m}+\mathcal{O}\left(\frac{1}{m^2}\right)\right)\left(1+\frac{1}{2m}+\mathcal{O}\left(\frac{1}{m^2}\right)\right)\left(1-\frac{1}{2m}+\mathcal{O}\left(\frac{1}{m^2}\right)\right)\\
&= \left(1-\frac{1}{2m}+\mathcal{O}\left(\frac{1}{m^2}\right)\right), \qquad (37)
\end{aligned}$$

for $3s+2 \in \{5k-3, 5k-2, \dots, 11k-4\}$,

$$\begin{aligned}
\widehat{w}_{m,1}^{3s+3} &= \left(1+\frac{b(3t+1)}{2bm(3t+1)+b(s-4t)+y_{-1}}\right)\left(1+\frac{-b(3t+1)}{2bm(3t+1)+b(s-2t+1)+y_{-2}}\right)\\
&\times \left(1+\frac{-b(3t+1)}{2bm(3t+1)+b(s+2)+y_{-3}}\right)\\
&= \left(1+\frac{1}{2m}+\mathcal{O}\left(\frac{1}{m^2}\right)\right)\left(1-\frac{1}{2m}+\mathcal{O}\left(\frac{1}{m^2}\right)\right)\left(1-\frac{1}{2m}+\mathcal{O}\left(\frac{1}{m^2}\right)\right)\\
&= \left(1-\frac{1}{2m}+\mathcal{O}\left(\frac{1}{m^2}\right)\right), \qquad (38)
\end{aligned}$$

for $3s+3 \in \{5k-3, 5k-2, \dots, 11k-4\}$,

$$\begin{aligned}
\widehat{w}_{m,1}^{3s+4} &= \left(1+\frac{-b(3t+1)}{2bm(3t+1)+b(s-2t+1)+y_{-1}}\right)\left(1+\frac{-b(3t+1)}{2bm(3t+1)+b(s+2)+y_{-2}}\right)\\
&\times \left(1+\frac{b(3t+1)}{2bm(3t+1)+b(s-4t+1)+y_{-3}}\right)\\
&= \left(1-\frac{1}{2m}+\mathcal{O}\left(\frac{1}{m^2}\right)\right)\left(1-\frac{1}{2m}+\mathcal{O}\left(\frac{1}{m^2}\right)\right)\left(1+\frac{1}{2m}+\mathcal{O}\left(\frac{1}{m^2}\right)\right)\\
&= \left(1-\frac{1}{2m}+\mathcal{O}\left(\frac{1}{m^2}\right)\right), \qquad (39)
\end{aligned}$$

for $3s+4 \in \{5k-3, 5k-2, \dots, 11k-4\}$,

$$\begin{aligned}
\widehat{w}_{m,2}^{3s+1} &= \left(1+\frac{b(3t+2)}{2bm(3t+2)+b(s-4t-2)+y_{-1}}\right)\left(1+\frac{-b(3t+2)}{2bm(3t+2)+b(s+1)+y_{-2}}\right)\\
&\times \left(1+\frac{-b(3t+2)}{2bm(3t+2)+b(s-2t)+y_{-3}}\right)\\
&= \left(1+\frac{1}{2m}+\mathcal{O}\left(\frac{1}{m^2}\right)\right)\left(1-\frac{1}{2m}+\mathcal{O}\left(\frac{1}{m^2}\right)\right)\left(1-\frac{1}{2m}+\mathcal{O}\left(\frac{1}{m^2}\right)\right)\\
&= \left(1-\frac{1}{2m}+\mathcal{O}\left(\frac{1}{m^2}\right)\right), \qquad (40)
\end{aligned}$$

for $3s+1 \in \{5k-3, 5k-2, \ldots, 11k-4\}$,

$$\begin{aligned}
\widehat{w}_{m,2}^{3s+2} &= \left(1+\frac{-b(3t+2)}{2bm(3t+2)+b(s+1)+y_{-1}}\right)\left(1+\frac{-b(3t+2)}{2bm(3t+2)+b(s-2t)+y_{-2}}\right)\\
&\times \left(1+\frac{b(3t+2)}{2bm(3t+2)+b(s-4t-1)+y_{-3}}\right)\\
&= \left(1-\frac{1}{2m}+\mathcal{O}\left(\frac{1}{m^2}\right)\right)\left(1-\frac{1}{2m}+\mathcal{O}\left(\frac{1}{m^2}\right)\right)\left(1+\frac{1}{2m}+\mathcal{O}\left(\frac{1}{m^2}\right)\right)\\
&= \left(1-\frac{1}{2m}+\mathcal{O}\left(\frac{1}{m^2}\right)\right), \qquad (41)
\end{aligned}$$

for $3s+2 \in \{5k-3, 5k-2, \ldots, 11k-4\}$,

$$\begin{aligned}
\widehat{w}_{m,2}^{3s+3} &= \left(1+\frac{-b(3t+2)}{2bm(3t+2)+b(s-2t)+y_{-1}}\right)\left(1+\frac{b(3t+2)}{2bm(3t+2)+b(s-4t-1)+y_{-2}}\right)\\
&\times \left(1+\frac{-b(3t+2)}{2bm(3t+2)+b(s+2)+y_{-3}}\right)\\
&= \left(1-\frac{1}{2m}+\mathcal{O}\left(\frac{1}{m^2}\right)\right)\left(1+\frac{1}{2m}+\mathcal{O}\left(\frac{1}{m^2}\right)\right)\left(1-\frac{1}{2m}+\mathcal{O}\left(\frac{1}{m^2}\right)\right)\\
&= \left(1-\frac{1}{2m}+\mathcal{O}\left(\frac{1}{m^2}\right)\right), \qquad (42)
\end{aligned}$$

for $3s+3 \in \{5k-3, 5k-2, \ldots, 11k-4\}$.
From (37)-(42), we have that the products in (17) are equiconvergent with the product,

$$\prod_{j=1}^{n}\left(1-\frac{1}{2j}+\mathcal{O}\left(\frac{1}{j^2}\right)\right)$$

that is with the sequence

$$exp\left(\sum_{j=1}^{n} ln\left(1-\frac{1}{2j}+\mathcal{O}\left(\frac{1}{j^2}\right)\right)\right) = exp\left(-\frac{1}{2}\sum_{j=1}^{n}\left(\frac{1}{j}+\mathcal{O}\left(\frac{1}{j^2}\right)\right)\right). \qquad (43)$$

From (43) and the fact that $\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{j}=\infty$, the statement follows. ∎

Now, we investigate the asymptotic behavior of solutions of Eq. (15) when $a=-1$, $b\neq 0$. From (16) by employing the following formulas

$$\begin{aligned}
x_{6(3t+1)m+3s+2} &= x_{3s+2-6(3t+1)}\prod_{p=0}^{m}\frac{(2y_{-2}-b)(-1)^{2p(3t+1)+s-t+1}+b}{(2y_{-1}-b)(-1)^{2p(3t+1)+s+1}+b}\\
&\times \frac{(2y_{-1}-b)(-1)^{2p(3t+1)+s-3t}+b}{(2y_{-3}-b)(-1)^{2p(3t+1)+s-2t+1}+b}\\
&\times \frac{(2y_{-3}-b)(-1)^{2p(3t+1)+s-5t}+b}{(2y_{-2}-b)(-1)^{2p(3t+1)+s-4t}+b},\ m\in\mathbb{N}_0, \qquad (44)
\end{aligned}$$

where $3s+2 \in \{5k-3, 5k-2, \dots, 11k-4\}$,

$$\begin{aligned} x_{6(3t+1)m+3s+3} &= x_{3s+3-6(3t+1)} \prod_{p=0}^{m} \frac{(2y_{-1}-b)(-1)^{2p(3t+1)+s-t+1}+b}{(2y_{-3}-b)(-1)^{2p(3t+1)+s+2}+b} \\ &\times \frac{(2y_{-3}-b)(-1)^{2p(3t+1)+s-3t+1}+b}{(2y_{-2}-b)(-1)^{2p(3t+1)+s-2t+1}+b} \\ &\times \frac{(2y_{-2}-b)(-1)^{2p(3t+1)+s-5t}+b}{(2y_{-1}-b)(-1)^{2p(3t+1)+s-4t}+b}, \ m \in \mathbb{N}_0, \end{aligned} \tag{45}$$

where $3s+3 \in \{5k-3, 5k-2, \dots, 11k-4\}$,

$$\begin{aligned} x_{6(3t+1)m+3s+4} &= x_{3s+4-6(3t+1)} \prod_{p=0}^{m} \frac{(2y_{-3}-b)(-1)^{2p(3t+1)+s-t+2}+b}{(2y_{-2}-b)(-1)^{2p(3t+1)+s+2}+b} \\ &\times \frac{(2y_{-2}-b)(-1)^{2p(3t+1)+s-3t+1}+b}{(2y_{-1}-b)(-1)^{2p(3t+1)+s-2t+1}+b} \\ &\times \frac{(2y_{-1}-b)(-1)^{2p(3t+1)+s-5t}+b}{(2y_{-3}-b)(-1)^{2p(3t+1)+s-4t+1}+b}, \ m \in \mathbb{N}_0, \end{aligned} \tag{46}$$

where $3s+4 \in \{5k-3, 5k-2, \dots, 11k-4\}$,

$$\begin{aligned} x_{6(3t+2)m+3s+1} &= x_{3s+1-6(3t+2)} \prod_{p=0}^{m} \frac{(2y_{-1}-b)(-1)^{2p(3t+2)+s-t}+b}{(2y_{-2}-b)(-1)^{2p(3t+2)+s+1}+b} \\ &\times \frac{(2y_{-2}-b)(-1)^{2p(3t+2)+s-3t-1}+b}{(2y_{-3}-b)(-1)^{2p(3t+2)+s-2t}+b} \\ &\times \frac{(2y_{-3}-b)(-1)^{2p(3t+2)+s-5t-2}+b}{(2y_{-1}-b)(-1)^{2p(3t+2)+s-4t-2}+b}, \ m \in \mathbb{N}_0, \end{aligned} \tag{47}$$

where $3s+1 \in \{5k-3, 5k-2, \dots, 11k-4\}$,

$$\begin{aligned} x_{6(3t+2)m+3s+2} &= x_{3s+2-6(3t+2)} \prod_{p=0}^{m} \frac{(2y_{-3}-b)(-1)^{2p(3t+2)+s-t+1}+b}{(2y_{-1}-b)(-1)^{2p(3t+2)+s+1}+b} \\ &\times \frac{(2y_{-1}-b)(-1)^{2p(3t+2)+s-3t-1}+b}{(2y_{-2}-b)(-1)^{2p(3t+2)+s-2t}+b} \\ &\times \frac{(2y_{-2}-b)(-1)^{2p(3t+2)+s-5t-2}+b}{(2y_{-3}-b)(-1)^{2p(3t+2)+s-4t-1}+b}, \ m \in \mathbb{N}_0, \end{aligned} \tag{48}$$

where $3s+2 \in \{5k-3, 5k-2, \dots, 11k-4\}$ and

$$\begin{aligned} x_{6(3t+2)m+3s+3} &= x_{3s+3-6(3t+2)} \prod_{p=0}^{m} \frac{(2y_{-2}-b)(-1)^{2p(3t+2)+s-t+1}+b}{(2y_{-3}-b)(-1)^{2p(3t+2)+s+2}+b} \\ &\times \frac{(2y_{-3}-b)(-1)^{2p(3t+2)+s-3t}+b}{(2y_{-1}-b)(-1)^{2p(3t+2)+s-2t}+b} \\ &\times \frac{(2y_{-1}-b)(-1)^{2p(3t+2)+s-5t-2}+b}{(2y_{-2}-b)(-1)^{2p(3t+2)+s-4t-1}+b}, \ m \in \mathbb{N}_0, \end{aligned} \tag{49}$$

where $3s+3 \in \{5k-3, 5k-2, \dots, 11k-4\}$.

Theorem 3.2. Suppose that $a=-1, b \neq 0$, $k=3t+r$, $r \in \{1,2\}$, $p \in \{1,5\}$, $i_1 \in \{1,2,\dots,6\}$, $i \in \{1,2,3\}$ and $t, v, h \in \mathbb{N}_0$. Let $N_i := \frac{b-y_i}{y_i}$. Then the following statements hold.

(a) If $y_{-1}=y_{-2}=y_{-3}=\frac{b}{2}$, then the sequence $(x_n)_{n\geq -k-3}$ is $6k-$periodic.

(b) If $y_{-2}=y_{-3}=\frac{b}{2}$ and $|N_1|<1$, then $x_{6(6h+2)m+6v+i_1}$, $x_{6(6h+4)m+6v+i_1+1}$ are constant and $\left|x_{6(6h+p)m+6v+2i}\right| \to \infty$, $x_{6(6h+1)m+6v+2i+1} \to 0$, $x_{6(6h+5)m+6v+2i-1} \to 0$, as $m \to \infty$.

(c) If $y_{-2}=y_{-3}=\frac{b}{2}$ and $|N_1|>1$, then $x_{6(6h+2)m+6v+i_1}$, $x_{6(6h+4)m+6v+i_1+1}$ are constant and $x_{6(6h+p)m+6v+2i} \to 0$, $\left|x_{6(6h+1)m+6v+2i+1}\right| \to \infty$, $\left|x_{6(6h+5)m+6v+2i-1}\right| \to \infty$, as $m \to \infty$.

(d) If $y_{-1}=y_{-3}=\frac{b}{2}$ and $|N_2|<1$, then $x_{6(6h+2)m+6v+i_1}$, $x_{6(6h+4)m+6v+i_1+1}$ are constant and $x_{6(6h+p)m+6v+2i} \to 0$, $\left|x_{6(6h+1)m+6v+2i+1}\right| \to \infty$, $\left|x_{6(6h+5)m+6v+2i-1}\right| \to \infty$, as $m \to \infty$.

(e) If $y_{-1}=y_{-3}=\frac{b}{2}$ and $|N_2|>1$, then $x_{6(6h+2)m+6v+i_1}$, $x_{6(6h+4)m+6v+i_1+1}$ are constant and $\left|x_{6(6h+p)m+6v+2i}\right| \to \infty$, $x_{6(6h+1)m+6v+2i+1} \to 0$, $x_{6(6h+5)m+6v+2i-1} \to 0$, as $m \to \infty$.

(f) If $y_{-1}=y_{-2}=\frac{b}{2}$ and $|N_3|<1$, then $x_{6(6h+2)m+6v+i_1}$, $x_{6(6h+4)m+6v+i_1+1}$ are constant and $\left|x_{6(6h+p)m+6v+2i}\right| \to \infty$, $x_{6(6h+1)m+6v+2i+1} \to 0$, $x_{6(6h+5)m+6v+2i-1} \to 0$, as $m \to \infty$.

(g) If $y_{-1}=y_{-2}=\frac{b}{2}$ and $|N_3|>1$, then $x_{6(6h+2)m+6v+i_1}$, $x_{6(6h+4)m+6v+i_1+1}$ are constant and $x_{6(6h+p)m+6v+2i} \to 0$, $\left|x_{6(6h+1)m+6v+2i+1}\right| \to \infty$, $\left|x_{6(6h+5)m+6v+2i-1}\right| \to \infty$, as $m \to \infty$.

(h) If $y_{-1}=\frac{b}{2}$, $y_{-2} \neq \frac{b+1}{2}$, $y_{-3} \neq \frac{b+1}{2}$, $y_{-2} \neq \frac{b}{2}$, $y_{-3} \neq \frac{b}{2}$ and $\left|\frac{N_2}{N_3}\right|<1$, then $x_{6(6h+2)m+6v+i_1}$, $x_{6(6h+4)m+6v+i_1+1}$ are constant and $x_{6(6h+p)m+6v+2i} \to 0$, $\left|x_{6(6h+1)m+6v+2i+1}\right| \to \infty$, $\left|x_{6(6h+5)m+6v+2i-1}\right| \to \infty$, as $m \to \infty$.

(i) If $y_{-1}=\frac{b}{2}$, $y_{-2} \neq \frac{b+1}{2}$, $y_{-3} \neq \frac{b+1}{2}$, $y_{-2} \neq \frac{b}{2}$, $y_{-3} \neq \frac{b}{2}$ and $\left|\frac{N_2}{N_3}\right|>1$, then $x_{6(6h+2)m+6v+i_1}$, $x_{6(6h+4)m+6v+i_1+1}$ are constant and $\left|x_{6(6h+p)m+6v+2i}\right| \to \infty$, $x_{6(6h+1)m+6v+2i+1} \to 0$, $x_{6(6h+5)m+6v+2i-1} \to 0$, as $m \to \infty$.

(j) If $y_{-1}=\frac{b}{2}$, $\frac{N_2}{N_3}=1$ then the sequence $(x_n)_{n\geq -k-3}$ is $6k-$periodic.

(k) If $y_{-1}=\frac{b}{2}$, $\frac{N_2}{N_3}=-1$ then the sequence $(x_n)_{n\geq -k-3}$ is $12k-$periodic.

(l) If $y_{-2}=\frac{b}{2}$, $y_{-1}\neq\frac{b+1}{2}$, $y_{-3}\neq\frac{b+1}{2}$, $y_{-1}\neq\frac{b}{2}$, $y_{-3}\neq\frac{b}{2}$ and $|N_1N_3|<1$, then $x_{6(6h+2)m+6v+i_1}$, $x_{6(6h+4)m+6v+i_1+1}$ are constant and $\left|x_{6(6h+p)m+6v+2i}\right|\to\infty$, $x_{6(6h+1)m+6v+2i+1}\to 0$, $x_{6(6h+5)m+6v+2i-1}\to 0$, as $m\to\infty$.

(m) If $y_{-2}=\frac{b}{2}$, $y_{-1}\neq\frac{b+1}{2}$, $y_{-3}\neq\frac{b+1}{2}$, $y_{-1}\neq\frac{b}{2}$, $y_{-3}\neq\frac{b}{2}$, and $|N_1N_3|>1$, then $x_{6(6h+2)m+6v+i_1}$, $x_{6(6h+4)m+6v+i_1+1}$ are constant and $x_{6(6h+p)m+6v+2i}\to 0$, $\left|x_{6(6h+1)m+6v+2i+1}\right|\to\infty$, $\left|x_{6(6h+5)m+6v+2i-1}\right|\to\infty$, as $m\to\infty$.

(n) If $y_{-2}=\frac{b}{2}$, $N_1N_3=1$ then the sequence $(x_n)_{n\geq -k-3}$ is $6k-$periodic.

(o) If $y_{-2}=\frac{b}{2}$, $N_1N_3=-1$ then the sequence $(x_n)_{n\geq -k-3}$ is $12k-$periodic.

(p) If $y_{-3}=\frac{b}{2}$, $y_{-1}\neq\frac{b+1}{2}$, $y_{-2}\neq\frac{b+1}{2}$, $y_{-1}\neq\frac{b}{2}$, $y_{-2}\neq\frac{b}{2}$ and $\left|\frac{N_1}{N_2}\right|<1$, then $x_{6(6h+2)m+6v+i_1}$, $x_{6(6h+4)m+6v+i_1+1}$ are constant and $\left|x_{6(6h+p)m+6v+2i}\right|\to\infty$, $x_{6(6h+1)m+6v+2i+1}\to 0$, $x_{6(6h+5)m+6v+2i-1}\to 0$, as $m\to\infty$.

(q) If $y_{-3}=\frac{b}{2}$, $y_{-1}\neq\frac{b+1}{2}$, $y_{-2}\neq\frac{b+1}{2}$, $y_{-1}\neq\frac{b}{2}$, $y_{-2}\neq\frac{b}{2}$, and $\left|\frac{N_1}{N_2}\right|>1$, then $x_{6(6h+2)m+6v+i_1}$, $x_{6(6h+4)m+6v+i_1+1}$ are constant and $x_{6(6h+p)m+6v+2i}\to 0$, $\left|x_{6(6h+1)m+6v+2i+1}\right|\to\infty$, $\left|x_{6(6h+5)m+6v+2i-1}\right|\to\infty$, as $m\to\infty$.

(r) If $y_{-3}=\frac{b}{2}$, $\frac{N_1}{N_2}=1$ then the sequence $(x_n)_{n\geq -k-3}$ is $6k-$periodic.

(s) If $y_{-3}=\frac{b}{2}$, $\frac{N_1}{N_2}=-1$ then the sequence $(x_n)_{n\geq -k-3}$ is $12k-$periodic.

Proof. (a): The result can be seen easily from Theorem 3.1 of item (i), .
Here, we will prove the items (b)-(c) since (d)-(e) and (f)-(g) can be proved similarly. Hence, proofs of items (d)-(e) and (f)-(g) are omitted.
(b)-(c): Assume that $y_{-2}=y_{-3}=\frac{b}{2}$ and $y_{-1}\neq\frac{b}{2}$. From (44)-(49), we have

$$\begin{aligned} x_{6(3t+1)m+3s+2} &= x_{3s+2-6(3t+1)}\prod_{p=0}^{m}\frac{(2y_{-1}-b)(-1)^{2p(3t+1)+s-3t}+b}{(2y_{-1}-b)(-1)^{2p(3t+1)+s+1}+b} \\ &= \frac{x_{3s+2-6(3t+1)}}{\left(\frac{(2y_{-1}-b)(-1)^{s+1}+b}{(2y_{-1}-b)(-1)^{s-3t}+b}\right)^{m+1}}, \end{aligned} \tag{50}$$

$$\begin{aligned} x_{6(3t+1)m+3s+3} &= x_{3s+3-6(3t+1)}\prod_{p=0}^{m}\frac{(2y_{-1}-b)(-1)^{2p(3t+1)+s-t+1}+b}{(2y_{-1}-b)(-1)^{2p(3t+1)+s-4t}+b} \\ &= \frac{x_{3s+3-6(3t+1)}}{\left(\frac{(2y_{-1}-b)(-1)^{s}+b}{(2y_{-1}-b)(-1)^{s-t+1}+b}\right)^{m+1}}, \end{aligned} \tag{51}$$

$$\begin{aligned} x_{6(3t+1)m+3s+4} &= x_{3s+4-6(3t+1)}\prod_{p=0}^{m}\frac{(2y_{-1}-b)(-1)^{2p(3t+1)+s-5t}+b}{(2y_{-1}-b)(-1)^{2p(3t+1)+s-2t+1}+b} \\ &= \frac{x_{3s+4-6(3t+1)}}{\left(\frac{(2y_{-1}-b)(-1)^{s+1}+b}{(2y_{-1}-b)(-1)^{s-5t}+b}\right)^{m+1}}, \end{aligned} \tag{52}$$

$$\begin{aligned} x_{6(3t+2)m+3s+1} &= x_{3s+1-6(3t+2)} \prod_{p=0}^{m} \frac{(2y_{-1}-b)(-1)^{2p(3t+2)+s-t}+b}{(2y_{-1}-b)(-1)^{2p(3t+2)+s-4t-2}+b} \\ &= \frac{x_{3s+1-6(3t+2)}}{\left(\frac{(2y_{-1}-b)(-1)^{s}+b}{(2y_{-1}-b)(-1)^{s-t}+b}\right)^{m+1}}, \end{aligned} \tag{53}$$

$$\begin{aligned} x_{6(3t+2)m+3s+2} &= x_{3s+2-6(3t+2)} \prod_{p=0}^{m} \frac{(2y_{-1}-b)(-1)^{2p(3t+2)+s-3t-1}+b}{(2y_{-1}-b)(-1)^{2p(3t+2)+s+1}+b} \\ &= \frac{x_{3s+2-6(3t+2)}}{\left(\frac{(2y_{-1}-b)(-1)^{s+1}+b}{(2y_{-1}-b)(-1)^{s-3t-1}+b}\right)^{m+1}}, \end{aligned} \tag{54}$$

and

$$\begin{aligned} x_{6(3t+2)m+3s+3} &= x_{3s+3-6(3t+2)} \prod_{p=0}^{m} \frac{(2y_{-1}-b)(-1)^{2p(3t+2)+s-5t-2}+b}{(2y_{-1}-b)(-1)^{2p(3t+2)+s-2t}+b} \\ &= \frac{x_{3s+3-6(3t+2)}}{\left(\frac{(2y_{-1}-b)(-1)^{s}+b}{(2y_{-1}-b)(-1)^{s-5t}+b}\right)^{m+1}}. \end{aligned} \tag{55}$$

From (50)-(55), there are four cases:

- **s and t are both even case:** $(s=2v,\ t=2h)$ We have

$$x_{6(6h+1)m+6v+2} = \frac{x_{6v+2-6(6h+1)}}{\left(\frac{(2y_{-1}-b)(-1)^{2v+1}+b}{(2y_{-1}-b)(-1)^{2v-6h}+b}\right)^{m+1}} = \frac{x_{6v+2-6(6h+1)}}{\left(\frac{b-y_{-1}}{y_{-1}}\right)^{m+1}}, \tag{56}$$

$$x_{6(6h+1)m+6v+3} = \frac{x_{6v+3-6(6h+1)}}{\left(\frac{(2y_{-1}-b)(-1)^{2v}+b}{(2y_{-1}-b)(-1)^{2v-2h+1}+b}\right)^{m+1}} = \frac{x_{6v+3-6(6h+1)}}{\left(\frac{y_{-1}}{b-y_{-1}}\right)^{m+1}}, \tag{57}$$

$$x_{6(6h+1)m+6v+4} = \frac{x_{6v+4-6(6h+1)}}{\left(\frac{(2y_{-1}-b)(-1)^{2v+1}+b}{(2y_{-1}-b)(-1)^{2v-10h}+b}\right)^{m+1}} = \frac{x_{6v+4-6(6h+1)}}{\left(\frac{b-y_{-1}}{y_{-1}}\right)^{m+1}}, \tag{58}$$

$$x_{6(6h+2)m+6v+1} = \frac{x_{6v+1-6(6h+2)}}{\left(\frac{(2y_{-1}-b)(-1)^{2v}+b}{(2y_{-1}-b)(-1)^{2v-2h}+b}\right)^{m+1}} = x_{6v+1-6(6h+2)}, \tag{59}$$

$$x_{6(6h+2)m+6v+2} = \frac{x_{6v+2-6(6h+2)}}{\left(\frac{(2y_{-1}-b)(-1)^{2v+1}+b}{(2y_{-1}-b)(-1)^{2v-6h-1}+b}\right)^{m+1}} = x_{6v+2-6(6h+2)}, \tag{60}$$

and

$$x_{6(6h+2)m+6v+3} = \frac{x_{6v+3-6(6h+2)}}{\left(\frac{(2y_{-1}-b)(-1)^{2v}+b}{(2y_{-1}-b)(-1)^{2v-10h}+b}\right)^{m+1}} = x_{6v+3-6(6h+2)}, \tag{61}$$

- **s is even and t is odd case:** $(s=2v,\ t=2h+1)$ We have

$$x_{6(6h+4)m+6v+2}=\frac{x_{6v+2-6(6h+4)}}{\left(\frac{(2y_{-1}-b)(-1)^{2v+1}+b}{(2y_{-1}-b)(-1)^{2v-6h-3}+b}\right)^{m+1}}=x_{6v+2-6(6h+4)}, \tag{62}$$

$$x_{6(6h+4)m+6v+3}=\frac{x_{6v+3-6(6h+4)}}{\left(\frac{(2y_{-1}-b)(-1)^{2v}+b}{(2y_{-1}-b)(-1)^{2v-2h}+b}\right)^{m+1}}=x_{6v+3-6(6h+4)}, \tag{63}$$

$$x_{6(6h+4)m+6v+4}=\frac{x_{6v+4-6(6h+4)}}{\left(\frac{(2y_{-1}-b)(-1)^{2v+1}+b}{(2y_{-1}-b)(-1)^{2v-10h-5}+b}\right)^{m+1}}=x_{6v+4-6(6h+4)}, \tag{64}$$

$$x_{6(6h+5)m+6v+1}=\frac{x_{6v+1-6(6h+5)}}{\left(\frac{(2y_{-1}-b)(-1)^{2v}+b}{(2y_{-1}-b)(-1)^{2v-2h-1}+b}\right)^{m+1}}=\frac{x_{6v+1-6(6h+5)}}{\left(\frac{y_{-1}}{b-y_{-1}}\right)^{m+1}}, \tag{65}$$

$$x_{6(6h+5)m+6v+2}=\frac{x_{6v+2-6(6h+5)}}{\left(\frac{(2y_{-1}-b)(-1)^{2v+1}+b}{(2y_{-1}-b)(-1)^{2v-6h-4}+b}\right)^{m+1}}=\frac{x_{6v+2-6(6h+5)}}{\left(\frac{b-y_{-1}}{y_{-1}}\right)^{m+1}}, \tag{66}$$

and

$$x_{6(6h+5)m+6v+3}=\frac{x_{6v+3-6(6h+5)}}{\left(\frac{(2y_{-1}-b)(-1)^{2v}+b}{(2y_{-1}-b)(-1)^{2v-10h-5}+b}\right)^{m+1}}=\frac{x_{6v+3-6(6h+5)}}{\left(\frac{y_{-1}}{b-y_{-1}}\right)^{m+1}}, \tag{67}$$

- **s is odd and t is even case:** $(s=2v+1,\ t=2h)$ We have

$$x_{6(6h+1)m+6v+5}=\frac{x_{6v+5-6(6h+1)}}{\left(\frac{(2y_{-1}-b)(-1)^{2v+2}+b}{(2y_{-1}-b)(-1)^{2v-6h+1}+b}\right)^{m+1}}=\frac{x_{6v+5-6(6h+1)}}{\left(\frac{y_{-1}}{b-y_{-1}}\right)^{m+1}}, \tag{68}$$

$$x_{6(6h+1)m+6v+6}=\frac{x_{6v+6-6(6h+1)}}{\left(\frac{(2y_{-1}-b)(-1)^{2v+1}+b}{(2y_{-1}-b)(-1)^{2v-2h+2}+b}\right)^{m+1}}=\frac{x_{6v+6-6(6h+1)}}{\left(\frac{b-y_{-1}}{y_{-1}}\right)^{m+1}}, \tag{69}$$

$$x_{6(6h+1)m+6v+7}=\frac{x_{6v+7-6(6h+1)}}{\left(\frac{(2y_{-1}-b)(-1)^{2v+2}+b}{(2y_{-1}-b)(-1)^{2v-10h+1}+b}\right)^{m+1}}=\frac{x_{6v+7-6(6h+1)}}{\left(\frac{y_{-1}}{b-y_{-1}}\right)^{m+1}}, \tag{70}$$

$$x_{6(6h+2)m+6v+4}=\frac{x_{6v+4-6(6h+2)}}{\left(\frac{(2y_{-1}-b)(-1)^{2v+1}+b}{(2y_{-1}-b)(-1)^{2v-2h+1}+b}\right)^{m+1}}=x_{6v+4-6(6h+2)}, \tag{71}$$

$$x_{6(6h+2)m+6v+5}=\frac{x_{6v+5-6(6h+2)}}{\left(\frac{(2y_{-1}-b)(-1)^{2v+2}+b}{(2y_{-1}-b)(-1)^{2v-6h}+b}\right)^{m+1}}=x_{6v+5-6(6h+2)}, \tag{72}$$

and

$$x_{6(6h+2)m+6v+6}=\frac{x_{6v+6-6(6h+2)}}{\left(\frac{(2y_{-1}-b)(-1)^{2v+1}+b}{(2y_{-1}-b)(-1)^{2v-10h+1}+b}\right)^{m+1}}=x_{6v+6-6(6h+2)}, \tag{73}$$

- s **and** t **are both odd case:** $(s=2v+1,\ t=2h+1)$ We have

$$x_{6(6h+4)m+6v+5}=\frac{x_{6v+5-6(6h+4)}}{\left(\frac{(2y_{-1}-b)(-1)^{2v+2}+b}{(2y_{-1}-b)(-1)^{2v-6h-2}+b}\right)^{m+1}}=x_{6v+5-6(6h+4)}, \tag{74}$$

$$x_{6(6h+4)m+6v+6}=\frac{x_{6v+6-6(6h+4)}}{\left(\frac{(2y_{-1}-b)(-1)^{2v+1}+b}{(2y_{-1}-b)(-1)^{2v-2h+1}+b}\right)^{m+1}}=x_{6v+6-6(6h+4)}, \tag{75}$$

$$x_{6(6h+4)m+6v+7}=\frac{x_{6v+7-6(6h+4)}}{\left(\frac{(2y_{-1}-b)(-1)^{2v+2}+b}{(2y_{-1}-b)(-1)^{2v-10h-4}+b}\right)^{m+1}}=x_{6v+7-6(6h+4)}, \tag{76}$$

$$x_{6(6h+5)m+6v+4}=\frac{x_{6v+4-6(6h+5)}}{\left(\frac{(2y_{-1}-b)(-1)^{2v+1}+b}{(2y_{-1}-b)(-1)^{2v-2h}+b}\right)^{m+1}}=\frac{x_{6v+4-6(6h+5)}}{\left(\frac{b-y_{-1}}{y_{-1}}\right)^{m+1}}, \tag{77}$$

$$x_{6(6h+5)m+6v+5}=\frac{x_{6v+5-6(6h+5)}}{\left(\frac{(2y_{-1}-b)(-1)^{2v+2}+b}{(2y_{-1}-b)(-1)^{2v-6h-3}+b}\right)^{m+1}}=\frac{x_{6v+5-6(6h+5)}}{\left(\frac{y_{-1}}{b-y_{-1}}\right)^{m+1}}, \tag{78}$$

and

$$x_{6(6h+5)m+6v+6}=\frac{x_{6v+6-6(6h+5)}}{\left(\frac{(2y_{-1}-b)(-1)^{2v+1}+b}{(2y_{-1}-b)(-1)^{2v-10h-4}+b}\right)^{m+1}}=\frac{x_{6v+6-6(6h+5)}}{\left(\frac{b-y_{-1}}{y_{-1}}\right)^{m+1}}. \tag{79}$$

From (56)-(79), the result can be seen easily.
(h)-(k): Assume that $y_{-1}=\frac{b}{2},\ y_{-2}\neq\frac{b+1}{2},\ y_{-3}\neq\frac{b+1}{2},\ y_{-2}\neq\frac{b}{2},\ y_{-3}\neq\frac{b}{2}$, and from (44)-(49) we have

$$\begin{aligned}
x_{6(3t+1)m+3s+2} &= x_{3s+2-6(3t+1)}\prod_{p=0}^{m}\frac{(2y_{-2}-b)(-1)^{2p(3t+1)+s-t+1}+b}{(2y_{-3}-b)(-1)^{2p(3t+1)+s-2t+1}+b}\\
&\times\frac{(2y_{-3}-b)(-1)^{2p(3t+1)+s-5t}+b}{(2y_{-2}-b)(-1)^{2p(3t+1)+s-4t}+b}\\
&=\frac{x_{3s+2-6(3t+1)}}{\left(\frac{(2y_{-3}-b)(-1)^{s+1}+b}{(2y_{-2}-b)(-1)^{s-t+1}+b}\right)^{m+1}\left(\frac{(2y_{-2}-b)(-1)^{s}+b}{(2y_{-3}-b)(-1)^{s-5t}+b}\right)^{m+1}},
\end{aligned} \tag{80}$$

$$\begin{aligned}
x_{6(3t+1)m+3s+3} &= x_{3s+3-6(3t+1)}\prod_{p=0}^{m}\frac{(2y_{-2}-b)(-1)^{2p(3t+1)+s-5t}+b}{(2y_{-3}-b)(-1)^{2p(3t+1)+s+2}+b}\\
&\times\frac{(2y_{-3}-b)(-1)^{2p(3t+1)+s-3t+1}+b}{(2y_{-2}-b)(-1)^{2p(3t+1)+s-2t+1}+b}\\
&=\frac{x_{3s+3-6(3t+1)}}{\left(\frac{(2y_{-3}-b)(-1)^{s}+b}{(2y_{-2}-b)(-1)^{s-5t}+b}\right)^{m+1}\left(\frac{(2y_{-2}-b)(-1)^{s+1}+b}{(2y_{-3}-b)(-1)^{s-3t+1}+b}\right)^{m+1}},
\end{aligned} \tag{81}$$

$$\begin{aligned} x_{6(3t+1)m+3s+4} &= x_{3s+4-6(3t+1)} \prod_{p=0}^{m} \frac{(2y_{-2}-b)(-1)^{2p(3t+1)+s-3t+1}+b}{(2y_{-3}-b)(-1)^{2p(3t+1)+s-4t+1}+b} \\ &\times \frac{(2y_{-3}-b)(-1)^{2p(3t+1)+s-t+2}+b}{(2y_{-2}-b)(-1)^{2p(3t+1)+s+2}+b} \\ &= \frac{x_{3s+4-6(3t+1)}}{\left(\frac{(2y_{-3}-b)(-1)^{s+1}+b}{(2y_{-2}-b)(-1)^{s-3t+1}+b}\right)^{m+1}\left(\frac{(2y_{-2}-b)(-1)^{s}+b}{(2y_{-3}-b)(-1)^{s-t}+b}\right)^{m+1}}, \end{aligned} \tag{82}$$

$$\begin{aligned} x_{6(3t+2)m+3s+1} &= x_{3s+1-6(3t+2)} \prod_{p=0}^{m} \frac{(2y_{-2}-b)(-1)^{2p(3t+2)+s-3t-1}+b}{(2y_{-3}-b)(-1)^{2p(3t+2)+s-2t}+b} \\ &\times \frac{(2y_{-3}-b)(-1)^{2p(3t+2)+s-5t-2}+b}{(2y_{-2}-b)(-1)^{2p(3t+2)+s+1}+b} \\ &= \frac{x_{3s+1-6(3t+2)}}{\left(\frac{(2y_{-3}-b)(-1)^{s}+b}{(2y_{-2}-b)(-1)^{s-3t-1}+b}\right)^{m+1}\left(\frac{(2y_{-2}-b)(-1)^{s+1}+b}{(2y_{-3}-b)(-1)^{s-5t}+b}\right)^{m+1}}, \end{aligned} \tag{83}$$

$$\begin{aligned} x_{6(3t+2)m+3s+2} &= x_{3s+2-6(3t+2)} \prod_{p=0}^{m} \frac{(2y_{-2}-b)(-1)^{2p(3t+2)+s-5t-2}+b}{(2y_{-3}-b)(-1)^{2p(3t+2)+s-4t-1}+b} \\ &\times \frac{(2y_{-3}-b)(-1)^{2p(3t+2)+s-t+1}+b}{(2y_{-2}-b)(-1)^{2p(3t+2)+s-2t}+b} \\ &= \frac{x_{3s+2-6(3t+2)}}{\left(\frac{(2y_{-3}-b)(-1)^{s-1}+b}{(2y_{-2}-b)(-1)^{s-5t}+b}\right)^{m+1}\left(\frac{(2y_{-2}-b)(-1)^{s}+b}{(2y_{-3}-b)(-1)^{s-t+1}+b}\right)^{m+1}}, \end{aligned} \tag{84}$$

and

$$\begin{aligned} x_{6(3t+2)m+3s+3} &= x_{3s+3-6(3t+2)} \prod_{p=0}^{m} \frac{(2y_{-2}-b)(-1)^{2p(3t+2)+s-t+1}+b}{(2y_{-3}-b)(-1)^{2p(3t+2)+s+2}+b} \\ &\times \frac{(2y_{-3}-b)(-1)^{2p(3t+2)+s-3t}+b}{(2y_{-2}-b)(-1)^{2p(3t+2)+s-4t-1}+b} \\ &= \frac{x_{3s+3-6(3t+2)}}{\left(\frac{(2y_{-3}-b)(-1)^{s}+b}{(2y_{-2}-b)(-1)^{s-t+1}+b}\right)^{m+1}\left(\frac{(2y_{-2}-b)(-1)^{s-1}+b}{(2y_{-3}-b)(-1)^{s-3t}+b}\right)^{m+1}}. \end{aligned} \tag{85}$$

From (80)-(85), there are four cases:

- s **and** t **are both even case:** $(s = 2v,\ t = 2h)$ We have

$$\begin{aligned} x_{6(6h+1)m+6v+2} &= \frac{x_{6v+2-6(6h+1)}}{\left(\frac{(2y_{-3}-b)(-1)^{2v+1}+b}{(2y_{-2}-b)(-1)^{2v-2h+1}+b}\right)^{m+1}\left(\frac{(2y_{-2}-b)(-1)^{2v}+b}{(2y_{-3}-b)(-1)^{2v-10h}+b}\right)^{m+1}} \\ &= \frac{x_{6v+2-6(6h+1)}}{\left(\frac{(b-y_{-3})y_{-2}}{(b-y_{-2})y_{-3}}\right)^{m+1}}, \end{aligned} \tag{86}$$

$$x_{6(6h+1)m+6v+3} = \frac{x_{6v+3-6(6h+1)}}{\left(\frac{(2y_{-3}-b)(-1)^{2v}+b}{(2y_{-2}-b)(-1)^{2v-10h}+b}\right)^{m+1}\left(\frac{(2y_{-2}-b)(-1)^{2v+1}+b}{(2y_{-3}-b)(-1)^{2v-6h+1}+b}\right)^{m+1}} = \frac{x_{6v+3-6(6h+1)}}{\left(\frac{y_{-3}(b-y_{-2})}{y_{-2}(b-y_{-3})}\right)^{m+1}}, \quad (87)$$

$$x_{6(6h+1)m+6v+4} = \frac{x_{6v+4-6(6h+1)}}{\left(\frac{(2y_{-3}-b)(-1)^{2v+1}+b}{(2y_{-2}-b)(-1)^{2v-6h+1}+b}\right)^{m+1}\left(\frac{(2y_{-2}-b)(-1)^{2v}+b}{(2y_{-3}-b)(-1)^{2v-2h}+b}\right)^{m+1}} = \frac{x_{6v+4-6(6h+1)}}{\left(\frac{(b-y_{-3})y_{-2}}{(b-y_{-2})y_{-3}}\right)^{m+1}}, \quad (88)$$

$$x_{6(6h+2)m+6v+1} = \frac{x_{6v+1-6(6h+2)}}{\left(\frac{(2y_{-3}-b)(-1)^{2v}+b}{(2y_{-2}-b)(-1)^{2v-6h-1}+b}\right)^{m+1}\left(\frac{(2y_{-2}-b)(-1)^{2v+1}+b}{(2y_{-3}-b)(-1)^{2v-10h}+b}\right)^{m+1}} = x_{6v+1-6(6h+2)}, \quad (89)$$

$$x_{6(6h+2)m+6v+2} = \frac{x_{6v+2-6(6h+2)}}{\left(\frac{(2y_{-3}-b)(-1)^{2v-1}+b}{(2y_{-2}-b)(-1)^{2v-10h}+b}\right)^{m+1}\left(\frac{(2y_{-2}-b)(-1)^{2v}+b}{(2y_{-3}-b)(-1)^{2v-2h+1}+b}\right)^{m+1}} = x_{6v+2-6(6h+2)}, \quad (90)$$

and

$$x_{6(6h+2)m+6v+3} = \frac{x_{6v+3-6(6h+2)}}{\left(\frac{(2y_{-3}-b)(-1)^{2v}+b}{(2y_{-2}-b)(-1)^{2v-2h+1}+b}\right)^{m+1}\left(\frac{(2y_{-2}-b)(-1)^{2v-1}+b}{(2y_{-3}-b)(-1)^{2v-6h}+b}\right)^{m+1}} = x_{6v+3-6(6h+2)}, \quad (91)$$

- **s is even and t is odd case:** $(s=2v,\ t=2h+1)$ We have

$$x_{6(6h+4)m+6v+2} = \frac{x_{6v+2-6(6h+4)}}{\left(\frac{(2y_{-3}-b)(-1)^{2v+1}+b}{(2y_{-2}-b)(-1)^{2v-2h}+b}\right)^{m+1}\left(\frac{(2y_{-2}-b)(-1)^{2v}+b}{(2y_{-3}-b)(-1)^{2v-10h-5}+b}\right)^{m+1}} = x_{6v+2-6(6h+4)}, \quad (92)$$

$$x_{6(6h+4)m+6v+3} = \frac{x_{6v+3-6(6h+4)}}{\left(\frac{(2y_{-3}-b)(-1)^{2v}+b}{(2y_{-2}-b)(-1)^{2v-10h-5}+b}\right)^{m+1}\left(\frac{(2y_{-2}-b)(-1)^{2v+1}+b}{(2y_{-3}-b)(-1)^{2v-6h-2}+b}\right)^{m+1}} = x_{6v+3-6(6h+4)}, \quad (93)$$

$$x_{6(6h+4)m+6v+4} = \frac{x_{6v+4-6(6h+4)}}{\left(\frac{(2y_{-3}-b)(-1)^{2v+1}+b}{(2y_{-2}-b)(-1)^{2v-6h-2}+b}\right)^{m+1}\left(\frac{(2y_{-2}-b)(-1)^{2v}+b}{(2y_{-3}-b)(-1)^{2v-2h-1}+b}\right)^{m+1}} = x_{6v+4-6(6h+4)}, \quad (94)$$

$$x_{6(6h+5)m+6v+1} = \frac{x_{6v+1-6(6h+5)}}{\left(\frac{(2y_{-3}-b)(-1)^{2v}+b}{(2y_{-2}-b)(-1)^{2v-6h-4}+b}\right)^{m+1}\left(\frac{(2y_{-2}-b)(-1)^{2v+1}+b}{(2y_{-3}-b)(-1)^{2v-10h-5}+b}\right)^{m+1}} = \frac{x_{6v+1-6(6h+5)}}{\left(\frac{y_{-3}(b-y_{-2})}{y_{-2}(b-y_{-3})}\right)^{m+1}}, \quad (95)$$

$$x_{6(6h+5)m+6v+2} = \frac{x_{6v+2-6(6h+5)}}{\left(\frac{(2y_{-3}-b)(-1)^{2v-1}+b}{(2y_{-2}-b)(-1)^{2v-10h-5}+b}\right)^{m+1}\left(\frac{(2y_{-2}-b)(-1)^{2v}+b}{(2y_{-3}-b)(-1)^{2v-2h}+b}\right)^{m+1}} = \frac{x_{6v+2-6(6h+5)}}{\left(\frac{(b-y_{-3})y_{-2}}{(b-y_{-2})y_{-3}}\right)^{m+1}}, \quad (96)$$

and

$$x_{6(6h+5)m+6v+3} = \frac{x_{6v+3-6(6h+5)}}{\left(\frac{(2y_{-3}-b)(-1)^{2v}+b}{(2y_{-2}-b)(-1)^{2v-2h}+b}\right)^{m+1}\left(\frac{(2y_{-2}-b)(-1)^{2v-1}+b}{(2y_{-3}-b)(-1)^{2v-6h-3}+b}\right)^{m+1}} = \frac{x_{6v+3-6(6h+5)}}{\left(\frac{y_{-3}(b-y_{-2})}{y_{-2}(b-y_{-3})}\right)^{m+1}}, \quad (97)$$

- **s is odd and t is even case:** $(s = 2v+1,\ t = 2h)$ We have

$$x_{6(6h+1)m+6v+5} = \frac{x_{6v+5-6(6h+1)}}{\left(\frac{(2y_{-3}-b)(-1)^{2v+2}+b}{(2y_{-2}-b)(-1)^{2v-2h+2}+b}\right)^{m+1}\left(\frac{(2y_{-2}-b)(-1)^{2v+1}+b}{(2y_{-3}-b)(-1)^{2v-10h+1}+b}\right)^{m+1}} = \frac{x_{6v+5-6(6h+1)}}{\left(\frac{y_{-3}(b-y_{-2})}{y_{-2}(b-y_{-3})}\right)^{m+1}}, \quad (98)$$

$$x_{6(6h+1)m+6v+6} = \frac{x_{6v+6-6(6h+1)}}{\left(\frac{(2y_{-3}-b)(-1)^{2v+1}+b}{(2y_{-2}-b)(-1)^{2v-10h+1}+b}\right)^{m+1}\left(\frac{(2y_{-2}-b)(-1)^{2v+2}+b}{(2y_{-3}-b)(-1)^{2v-6h+2}+b}\right)^{m+1}} = \frac{x_{6v+6-6(6h+1)}}{\left(\frac{(b-y_{-3})y_{-2}}{(b-y_{-2})y_{-3}}\right)^{m+1}}, \quad (99)$$

$$x_{6(6h+1)m+6v+7} = \frac{x_{6v+7-6(6h+1)}}{\left(\frac{(2y_{-3}-b)(-1)^{2v+2}+b}{(2y_{-2}-b)(-1)^{2v-6h+2}+b}\right)^{m+1}\left(\frac{(2y_{-2}-b)(-1)^{2v+1}+b}{(2y_{-3}-b)(-1)^{2v-2h+1}+b}\right)^{m+1}} = \frac{x_{6v+7-6(6h+1)}}{\left(\frac{y_{-3}(b-y_{-2})}{y_{-2}(b-y_{-3})}\right)^{m+1}}, \quad (100)$$

$$x_{6(6h+2)m+6v+4} = \frac{x_{6v+4-6(6h+2)}}{\left(\frac{(2y_{-3}-b)(-1)^{2v+1}+b}{(2y_{-2}-b)(-1)^{2v-6h}+b}\right)^{m+1}\left(\frac{(2y_{-2}-b)(-1)^{2v+2}+b}{(2y_{-3}-b)(-1)^{2v-10h+1}+b}\right)^{m+1}} = x_{6v+4-6(6h+2)}, \quad (101)$$

$$x_{6(6h+2)m+6v+5} = \frac{x_{6v+5-6(6h+2)}}{\left(\frac{(2y_{-3}-b)(-1)^{2v}+b}{(2y_{-2}-b)(-1)^{2v-10h+1}+b}\right)^{m+1}\left(\frac{(2y_{-2}-b)(-1)^{2v+1}+b}{(2y_{-3}-b)(-1)^{2v-2h+2}+b}\right)^{m+1}} = x_{6v+5-6(6h+2)}, \tag{102}$$

and

$$x_{6(6h+2)m+6v+6} = \frac{x_{6v+6-6(6h+2)}}{\left(\frac{(2y_{-3}-b)(-1)^{2v+1}+b}{(2y_{-2}-b)(-1)^{2v-2h}+b}\right)^{m+1}\left(\frac{(2y_{-2}-b)(-1)^{2v}+b}{(2y_{-3}-b)(-1)^{2v-6h+1}+b}\right)^{m+1}} = x_{6v+6-6(6h+2)}, \tag{103}$$

- ***s* and *t* are both odd case:** $(s=2v+1,\ t=2h+1)$ We have

$$x_{6(6h+4)m+6v+5} = \frac{x_{6v+5-6(6h+4)}}{\left(\frac{(2y_{-3}-b)(-1)^{2v+2}+b}{(2y_{-2}-b)(-1)^{2v-2h+1}+b}\right)^{m+1}\left(\frac{(2y_{-2}-b)(-1)^{2v+1}+b}{(2y_{-3}-b)(-1)^{2v-10h-4}+b}\right)^{m+1}} = x_{6v+5-6(6h+4)}, \tag{104}$$

$$x_{6(6h+4)m+6v+6} = \frac{x_{6v+6-6(6h+4)}}{\left(\frac{(2y_{-3}-b)(-1)^{2v+1}+b}{(2y_{-2}-b)(-1)^{2v-10h-4}+b}\right)^{m+1}\left(\frac{(2y_{-2}-b)(-1)^{2v+2}+b}{(2y_{-3}-b)(-1)^{2v-6h-1}+b}\right)^{m+1}} = x_{6v+6-6(6h+4)}, \tag{105}$$

$$x_{6(6h+4)m+6v+7} = \frac{x_{6v+7-6(6h+4)}}{\left(\frac{(2y_{-3}-b)(-1)^{2v+2}+b}{(2y_{-2}-b)(-1)^{2v-6h-1}+b}\right)^{m+1}\left(\frac{(2y_{-2}-b)(-1)^{2v+1}+b}{(2y_{-3}-b)(-1)^{2v-2h}+b}\right)^{m+1}} = x_{6v+7-6(6h+4)}, \tag{106}$$

$$x_{6(6h+5)m+6v+4} = \frac{x_{6v+4-6(6h+5)}}{\left(\frac{(2y_{-3}-b)(-1)^{2v+1}+b}{(2y_{-2}-b)(-1)^{2v-6h-3}+b}\right)^{m+1}\left(\frac{(2y_{-2}-b)(-1)^{2v+2}+b}{(2y_{-3}-b)(-1)^{2v-10h-4}+b}\right)^{m+1}} = \frac{x_{6v+4-6(6h+5)}}{\left(\frac{(b-y_{-3})y_{-2}}{(b-y_{-2})y_{-3}}\right)^{m+1}}, \tag{107}$$

$$x_{6(6h+5)m+6v+5} = \frac{x_{6v+5-6(6h+5)}}{\left(\frac{(2y_{-3}-b)(-1)^{2v}+b}{(2y_{-2}-b)(-1)^{2v-10h-4}+b}\right)^{m+1}\left(\frac{(2y_{-2}-b)(-1)^{2v+1}+b}{(2y_{-3}-b)(-1)^{2v-2h+1}+b}\right)^{m+1}} = \frac{x_{6v+5-6(6h+5)}}{\left(\frac{y_{-3}(b-y_{-2})}{y_{-2}(b-y_{-3})}\right)^{m+1}}, \tag{108}$$

and

$$x_{6(6h+5)m+6v+6} = \frac{x_{6v+6-6(6h+5)}}{\left(\frac{(2y_{-3}-b)(-1)^{2v+1}+b}{(2y_{-2}-b)(-1)^{2v-2h+1}+b}\right)^{m+1}\left(\frac{(2y_{-2}-b)(-1)^{2v}+b}{(2y_{-3}-b)(-1)^{2v-6h-2}+b}\right)^{m+1}} = \frac{x_{6v+6-6(6h+5)}}{\left(\frac{(b-y_{-3})y_{-2}}{(b-y_{-2})y_{-3}}\right)^{m+1}}, \tag{109}$$

From (86)-(109), the result can be seen easily.
The proofs of (l)-(o) and (p)-(s) can be proved similarly to proofs of to (k)-(l) and are omitted. ∎

Finally we investigate the asymptotic behavior of solutions of Eqs.(16) and Eqs. (17) when $a \neq 0, b = 0$, by employing the following formulas, for the case $a \neq 1$,

$$x_{6(3t+r)m+3s+j_1} = x_{3s+j_1-6(3t+r)} \prod_{p=0}^{m} \frac{1}{a^{3t+r}},\ m \in \mathbb{N}_0, \tag{110}$$

while for $a = 1$,

$$x_{6(3t+r)m+3s+j_1} = x_{3s+j_1-6(3t+r)},\ m \in \mathbb{N}_0, \tag{111}$$

By using above formulas, we give the following theorem. Proof of the theorem can be seen easily from (110)-(111). So we will omit the proof of the following theorem.

Theorem 3.3. Suppose that that $a \neq 0$, $b = 0$, $k = 3t + r$, $r \in \{1, 2\}$, $3s + j_1 \in \{5k - 3, 5k - 2, \dots, 11k - 4\}$ and $t \in \mathbb{N}_0$. Then the next statements hold.

(a) If $|a| > 1$, then $x_m \to 0$, as $m \to \infty$.

(b) If $|a| < 1$, then $|x_m| \to \infty$, as $m \to \infty$.

(c) If $a = 1$, then the sequence $(x_n)_{n \geq -k-3}$ is $6k-$periodic.

(d) If $a = -1$, then the sequence $(x_n)_{n \geq -k-3}$ is $12k-$periodic.

4 Conclusion

In this study, we have investigated the following difference equation

$$x_n = \frac{x_{n-k} x_{n-k-l}}{x_{n-l}\left(a_n + b_n x_{n-k} x_{n-k-l}\right)},\ n \in \mathbb{N}_0,$$

where $k, l \in \mathbb{N}$, $(a_n)_{n\in\mathbb{N}_0}, (b_n)_{n\in\mathbb{N}_0}$ and the initial values x_{-i}, $i = \overline{1, k+l}$, are real numbers.

Firstly, we have obtained the closed form of well defined solutions of the aforementioned equation using suitable transformation. In addition, we describe the forbidden set of the initial values using the obtained formulas. Finally, in the case where the coeffcients are constant and $k = 3$, $l = k$ in the equation, we have examined the asymptotic behavior and the periodicity of the solutions of this equation.

The equation (1) can extend to the $p-$dimensional system of difference equations which is variable coefficients or constant coefficients.

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